Circular Geodesic Submanifolds with Parallel Mean Curvature Vector in a Non-flat Complex Space Form

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1. Introduction. Let $f: M \to \tilde{M}$ be an isometric immersion of a connected complete Riemannian manifold M into a Riemannian manifold $\tilde{M}$. We call M a *circular geodesic* submanifold of $\tilde{M}$ provided that for every geodesic γ of M the curve $f(\gamma)$ is a circle in $\tilde{M}$. It is well-known that a round sphere is the only circular geodesic surface in $\boldsymbol{R}^3$. This result is generalized as follows: M^n is a circular geodesic submanifold of a real space form $\tilde{M}^{n+p}(c)$ of curvature c (that is, $\tilde{M}^{n+p}(c) = \boldsymbol{R}^{n+p}$, $S^{n+p}(c)$ or $\boldsymbol{RH}^{n+p}(c)$) if and only if M^n is totally umbilic in $\tilde{M}^{n+p}(c)$ or M^n is locally congruent to one of the compact symmetric spaces of rank one which is immersed into $\tilde{M}^{n+p}(c)$ with parallel second fundamental form (see, [8]).

In this paper, we consider the classification problem of circular geodesic submanifolds in a complex space form $\tilde{M}^N(c)$ of constant holomorphic sectional curvature c (that is, $\tilde{M}^N(c) = \boldsymbol{C}^N$, $\boldsymbol{CP}^N(c)$ or $\boldsymbol{CH}^N(c)$). The classification problem of circular geodesic submanifolds in a non-flat complex space form $\tilde{M}^N(c)$ is still open. In a complex space form $\tilde{M}^N(c)$, all examples of circular geodesic submanifolds what we know are parallel submanifolds (for details, see [4]). Needless to say, a parallel submanifold is not necessarily circular geodesic. The classification problem of parallel submanifolds in a non-flat complex space form was solved by Nakagawa, Naitoh, and Takagi ([5] and [6]).

Along this context, it is natural to consider the problem "In a complex space form $\tilde{M}^N(c)$ $(c \neq 0)$, does a circular geodesic submanifold have parallel second fundamental form?". We here give an affirmative partial answer to this problem. The main purpose of this paper is to prove the following.

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Theorem. *Let M be a circular geodesic submanifold of a non-flat complex space form $\tilde{M}^N(c)$. Suppose that the mean curvature vector of M is parallel with respect to the normal connection on the normal bundle $T^\perp M$ of M in $\tilde{M}^N(c)$. Then M has parallel second fundamental form in $\tilde{M}^N(c)$.*

2. Preliminaries. First we recall the notion of circles in a Riemannian manifold $\tilde{M}$. A curve $\gamma(s)$ of $\tilde{M}$ parametrized by arclength s is called a *circle* of curvature k, if there exists a field of unit vectors Y_s along the curve γ which satisfies the differential equations: $\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = kY_s$ and $\tilde{\nabla}_{\dot{\gamma}}Y_s = -k\dot{\gamma}$, where k is a positive constant and $\tilde{\nabla}_{\dot{\gamma}}$ denotes the covariant differentiation $\tilde{\nabla}$ along γ. Next we review the notion of CR-submanifolds of a Kaehler manifold. A Riemannian submanifold M of a Kaehler manifold $\tilde{M}$ with complex structure J is called a *CR-submanifold* if there exists on M a C^∞-holomorphic distribution $\mathfrak{D}$ satisfying its orthogonal complement $\mathfrak{D}^\perp$ is a totally real distribution, i.e., $J\mathfrak{D}_p^\perp \subseteq T_p^\perp(M)$ for any $p \in M$. We note that all holomorphic submanifolds, totally real submanifolds and real hypersurfaces are necessarily CR-submanifolds. The manifold M is said to be a *λ-isotropic* submanifold of $\tilde{M}$ provided that $\| \sigma(X, X) \|$ is equal to a constant $(= \lambda)$ for all unit tangent vectors X at its each point, where σ is the second fundamental form of M in $\tilde{M}$ ([7]). In particular, the function λ is constant on M, the immersion is said to be $(\lambda -)$ *constant isotropic*. The notion of isotropic is a generalization of "totally umbilic". We remark that these two definitions are coincidental, when *codim* $M = 1$.

We now prepare the following three lemmas without proof in order to prove our Theorem:

Lemma 1 ([2]). *Let M be a Riemannian submanifold of $\tilde{M}$. Then the following two conditions are equivalent:*

(i) *M is a circular geodesic submanifold of $\tilde{M}$.*

(ii) *The submanifold M is nonzero constant isotropic and the second fundamental form σ of M in $\tilde{M}$ is* *cyclic parallel, that is, σ satisfies*

$$(\bar{\nabla}_X\sigma)(Y, Z) + (\bar{\nabla}_Y\sigma)(Z, X) + (\bar{\nabla}_Z\sigma)(X, Y) = 0$$

for all vecter fields X, Y, and Z on M, where $\bar{\nabla}$ is the covariant differentiation of the second fundamental form σ.

Let M be an n-dimensional Riemannian submanifold in an N-dimensional complex space form (with complex structure J) $\tilde{M}^N(c)$ of constant holomorphic sectional curvature c. We here write the equation of Codazzi for M in $\tilde{M}^N(c)$:

$$(2.1)\quad (c/4)\{\langle JY, Z\rangle JX - \langle JX, Z\rangle JY + 2\langle X, JY\rangle JZ\}^\perp = (\bar{\nabla}_X\sigma)(Y, Z) - (\bar{\nabla}_Y\sigma)(X, Z),$$

where $\{*\}^\perp$ means the normal component of $\{*\}$. From 1 and (2.1) we get

Lemma 2 ([2]). *Let M be a Riemannian submanifold in a complex space form $\tilde{M}^N(c)$ of constant holomorphic sectional curvature c with complex structure J. Then the following are equivalent:*
(i) *The second fundamental form σ of M in $\tilde{M}^N(c)$ is cyclic parallel.*
(ii) $(\bar{\nabla}_X\sigma)(Y, Z) = (c/4)\{\langle X, JY\rangle JZ + \langle X, JZ\rangle JY\}^\perp$ *for all vector fields X, Y and Z tangent to M.*

The following is a key lemma for our Theorem.

Lemma 3 ([2]). *Let M be a circular geodesic CR-submanifold in a complex space form $\tilde{M}^N(c)$. Then the second fundamental form σ of M in $\tilde{M}^N(c)$ is parallel.*

3. Proof of theorem. We have only to show the following

Lemma 4. *Let M be an n-dimensional Riemannian submanifold with parallel mean curvature vector in a complex space form $\tilde{M}^N(c)$. Suppose that the second fundamental form of M in $\tilde{M}^N(c)$ is cyclic parallel. Then M is a CR-submanifold of $\tilde{M}^N(c)$. Moreover, for each $x \in M$, $T_x^\perp(M)$ is decomposed as: $T_x^\perp(M) = V_x \oplus V_x^\perp$, where $JV_x = V_x$, $V_x^\perp = J\mathfrak{D}_x^\perp$ and $\mathfrak{D}^\perp : p \to \mathfrak{D}_p^\perp$ is the totally distribution of M.*

Proof. First of all we define the tensor $\phi: TM \to TM$ as $\phi(X) = (JX)^T$ for any $X \in TM$, where $(*)^T$ is the tangential component of $(*)$. We set $U = \phi(TM)$. For an arbitrary fixed point $x \in M$, let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $T_x(M)$. Then $U_x = Span\{(Je_1)^T, (Je_2)^T, \cdots, (Je_n)^T\}$. It follows from the statement (ii) in Lemma 2 that

$$(3.1)\ 0 = \sum_{i=1}^n (\bar{\nabla}_X\sigma)(e_i, e_i) = 2\sum_{i=1}^n \langle X, Je_i\rangle (Je_i)^\perp.$$

Here and in the following we choose an orthonormal basis $\{e_1, \cdots, e_p, \cdots, e_n\}$ of $T_x(M)$ in such a way that $U_x = Span\{e_1, \cdots, e_p\}$. We consider the decomposition of $T_x(M)$ as: $T_x(M) = U_x \oplus U_x^\perp$, where $U_x^\perp$ is the orthogonal complement of U_x in $T_x(M)$. We shall show that $JU_x = U_x$. From (3.1) we have

$$0 = \sum_{i=1}^n \langle X, Je_i\rangle (Je_i)^\perp = -\sum_{i=1}^n \langle (JX)^T, e_i\rangle (Je_i)^\perp = -\sum_{a=1}^p \langle (JX)^T, e_a\rangle (Je_a)^\perp.$$

For any $e_a (a = 1, \cdots, p)$ we can choose $X(\in U_x)$ with $(JX)^T = e_a$. Then the above equation yields $(Je_a)^\perp = 0 (a = 1, \cdots, p)$ so that $JU_x = U_x$. Next, for $e_r (r = p + 1, \cdots, n)$ and $e_i (i = 1, \cdots, n)$ we find that $0 = \langle Je_i, e_r\rangle = -\langle e_i, Je_r\rangle$, which implies that $JU_x^\perp \subseteq T_x^\perp(M)$ so that $dim\ U_x$ is constant for any $x \in M$. Therefore our manifold is a CR-submanifold of $\tilde{M}^N(c)$. We shall show the latter half of Lemma 4. We decompose $T_x^\perp(M)$ as: $T_x^\perp(M) = (JU_x^\perp)^\perp \oplus JU_x^\perp$, where $(JU_x^\perp)^\perp$ is the orthogonal complement of $JU_x^\perp$ in $T_x^\perp(M)$. We set $V_x = (JU_x^\perp)^\perp$ so that $V_x^\perp = JU_x^\perp$, where $V_x^\perp$ is the orthogonal complement of V_x in $T_x^\perp(M)$. By the above discussion for any $X \in V_x$ we know that JX is perpendicular to $V_x^\perp$ and $T_x(M) (= U_x \oplus U_x^\perp)$ so that $JV_x = V_x$. Thus we get the conclusion.

By virtue of Lemma 1, Lemma 3, and Lemma 4 we obtain our Theorem.

4. Problem. Motivated by our Theorem we pose the following.

Problem. *Let M be a circular geodesic submanifold of a non-flat complex space form $\tilde{M}^N(c)$. If the length of the mean curvature vetor of M in $\tilde{M}^N(c)$ is constant, is the second fundamental form of M in $\tilde{M}^N(c)$ parallel?*

When $dim\ M = 2$, this problem was solved affirmatively (see, [3]).

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