On Standard L-Functions for Unitary Groups*)

By Keiji TAKANO

Department of Mathematics, Osaka University

(Communicated by Shokichi IYANAGA, M. J. A., Jan. 13, 1997)

Introduction. Let $G_n = U_{n,n}(\boldsymbol{K}/k)$ be the quasi-split unitary group of $2n$-dimension with respect to a quadratic extension $\boldsymbol{K}/k$ of number fields. The basic identity of Rankin-Selberg integral established in [4], [5] interpolates the *standard* L-function of a cuspidal automorphic representation of $G_n(\boldsymbol{A})$. In [5], the two of main steps in the theory of Rankin-Selberg method were carried out, though the group was $G_n = Sp_n$, or $O_{n,n}$;

(1) the investigation of analytic properties of the global Rankin-Selberg integral and,

(2) the computations of unramified local integrals.

These two parts can be carried out entirely in the same way as [5] also for our group $G_n = U_{n,n}(\boldsymbol{K}/k)$, which we shall state in §1.

The main part of this paper is devoted to the study of local integrals including finite ramified and archimedean places. We shall extend the method of [5] to adapt to representations that cannot be embedded in principal series representations. We rewrite these integrals by the Godement-Jacquet zeta integrals and obtain the analytic continuations of them. Then it is seen that, at finite ramified places, they can be made constant for a suitable choice of a test function, which enables us to prove the finiteness of poles of the partial standard L-function by the usual procedure of the Rankin-Selberg method.

The author would like to thank Takao Watanabe for many helpful advices and encouragement.

Notation. Let k be a number field and k_v be the completion of k at a place v of k. Let $G_n = U_{n,n}(\boldsymbol{K}/k)$ be the quasi-split form of unitary group of $2n$-dimension defined with respect to a quadratic extension $\boldsymbol{K}/k$ of number fields. The Galois involution of $\boldsymbol{K}/k$ is denoted by $x \mapsto \bar{x}$. We realize the group of k-points of G_n as

$$G_n(k) = \{g \in GL_{2n}(\boldsymbol{K}) \mid gJ_n{}^t\bar{g} = J_n\},$$
$$\text{where } J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

We often write simply as $G_n = G_n(k)$, if there is no fear of confusion. If a place v splits (resp. remains prime) in $\boldsymbol{K}$, $G_n(k_v)$ is isomorphic to $GL_{2n}(k_v)$ (resp. $U_{n,n}(\boldsymbol{K}_v/k_v)$ where $\boldsymbol{K}_v = \boldsymbol{K} \otimes_k k_v$ is a quadratic field extension of k_v). Let $K_{n,v}$ be the standard maximal compact subgroup of $G_n(k_v)$.

Let T_n (resp. A_n) be the maximal k-torus (resp. maximal k-split torus) given by

$$T_n = \{\mathrm{diag}(t_1, \dots, t_n, \bar{t}_1^{-1}, \dots, \bar{t}_n^{-1}) \mid t_i \in \boldsymbol{K}^\times\},$$
$$A_n = \{\mathrm{diag}(a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}) \mid a_i \in k^\times\}$$

and let χ_i be the k-rational character of A_n defined by $\chi_i(\mathrm{diag}(a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1})) = a_i$ for $1 \leqslant i \leqslant n$. Then $\{\chi_1, \dots, \chi_n\}$ forms a $\boldsymbol{Z}$-basis of $X^*(A_n) = \mathrm{Hom}(A_n, \boldsymbol{G}_m)$. Let $B_n = T_n \ltimes N_n$ be the Borel subgroup of G_n, of which the unipotent radical N_n is the subgroup consists of elements of the form $\begin{pmatrix} u & x \\ 0 & {}^t\bar{u}^{-1} \end{pmatrix}$, where $u \in GL_n(\boldsymbol{K})$ is upper triangular with ones in diagonals, and $x \in \mathrm{Mat}_n(\boldsymbol{K})$ is such that $x = {}^t\bar{x}$. Let $\Phi_n = \Phi(G_n, A_n)$ be the relative root system of G_n with respect to A_n and let Φ_n^+ be the set of positive roots corresponding to B_n explicitly given by $\Phi_n^+ = \{2\chi_i (1 \leqslant i \leqslant n), \chi_i \pm \chi_j (1 \leqslant i < j \leqslant n)\}$. Denote by $W_n = W(G_n, A_n)$ the relative Weyl group of G_n. For each $\alpha \in \Phi$, let $N_{n,\alpha}$ be the root subgroup determined by α.

For each integer r with $1 \leqslant r \leqslant n$, let $P_n^{(r)} = M_n^{(r)} \ltimes U_n^{(r)}$ be the maximal parabolic k-subgroup of G_n given by

$$M_n^{(r)} = \left\{ \iota\left(x, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \left(\begin{array}{cc|cc} x & & & \\ & A & & B \\ \hline & & {}^t\bar{x}^{-1} & \\ & C & & D \end{array} \right) \middle| \; x \in GL_r(\boldsymbol{K}), \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_{n-r} \right\}$$

---

*) Partly supported by Reseach Fellowship of the Japan Society for the promotion of Science for Young Scientists.

$$U_n^{(r)} = \left\{ \begin{pmatrix} 1_r & * & * & * \\ & 1_{n-r} & * & 0 \\ & & 1_r & 0 \\ & & * & 1_{n-r} \end{pmatrix} \in N_n \right\}.$$

Every proper parabolic k-subgroup of G_n is contained in a conjugate of some $P_n^{(r)}$.

The Galois form of the L-group of G_n is $GL_{2n}(\boldsymbol{C}) \rtimes \mathrm{Gal}(\boldsymbol{K}/k)$, where the non-trivial element of $\mathrm{Gal}(\boldsymbol{K}/k)$ acts on $GL_{2n}(\boldsymbol{C})$ by $g \mapsto J_n {}^t g^{-1} J_n^{-1}$. Now, let r_{st} be the $4n$-dimensional representation of ${}^L G_n$ induced from the standard $2n$-dimensional representation of the connected component $GL_{2n}(\boldsymbol{C})$ of ${}^L G_n$. Then, for a cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $G_n(\boldsymbol{A})$, the L-factor $L_{\mathrm{st}}(s, \pi_v) = L(s, \pi_v, r_{\mathrm{st}})$ is defined for any unramified places v (see [1]). The automorphic L-function we shall consider is a *partial* one, defined by the infinite product

$$L_{\mathrm{st}}(s, \pi) = \prod_{v;\ \mathrm{unramified}} L_{\mathrm{st}}(s, \pi_v),$$

which converges for $Re(s)$ large.

1. Review of global theory. Let $G_n = U_{n,n}(\boldsymbol{K}/k)$ act on a $2n$-demensional non-degenerate skew-hermitian space (V, ϕ) over $\boldsymbol{K}$ as the isometry group. Then the space $(V \oplus V, \phi \oplus (-\phi))$ also is skew-hermitian and non-degenerate, of which the isometry group can be identified with G_{2n}. In this way have an embedding $i : G_n \times G_n \hookrightarrow G_{2n}$. Let P be the stabilizer of the isotropic subspace $V^d = \{(v, v) \in V \oplus V ; v \in V\}$ of $V \oplus V$ in G_{2n}. It is a 'Siegel-type' maximal parabolic subgroup of G_{2n}, whose Levi part is isomorphic to $GL_{2n}(\boldsymbol{K})$.

Let $\delta_{P(\boldsymbol{A})}$ be the module of $P(\boldsymbol{A})$ and for each $s \in \boldsymbol{C}$, set $\delta_s(\cdot) = (\delta_{P(\boldsymbol{A})}(\cdot))^{s/2n}$. Let $\mathscr{T}(s) = \mathrm{ind}_{P(\boldsymbol{A})}^{G_{2n}(\boldsymbol{A})}(\delta_s)$ be the unnormalized induction. For $f_s \in \mathscr{T}(s)$ and $h \in G_{2n}(\boldsymbol{A})$, define

$$E(h; f_s) = \sum_{\gamma \in P(k) \backslash G_{2n}(k)} f_s(\gamma h).$$

The right hand side is known to converge absolutely for $Re(s)$ sufficiently large, uniformly for h in a fixed compact set. It can be continued analytically to a meromorphic function on whole of $s \in \boldsymbol{C}$. Following the method of [5,§5], put

$$d(s) = \prod_{l=1}^{2n} \zeta_k(2s - 2l + 2) \prod_{l=1}^{n} \zeta_{\boldsymbol{K}}(2s - 2l + 1)$$

where ζ_k (resp. $\zeta_{\boldsymbol{K}}$) is the completed Dedekind ζ-function of k (resp. $\boldsymbol{K}$). ($d(s)$ is chosen so that it cancels all the "denominators" of c-functions appearing in the constant term of $E(h; f_s)$). Then the number of poles of $d(s) \times E(h; f_s)$ are finite and located at integral and half integral points in the interval $[0,4n]$. (The occurence of poles depends on the choice of f_s).

Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $G_n(\boldsymbol{A})$, $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ be the contragradient of π, $\varphi = \otimes_v \varphi_v \in \pi$, $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v \in \tilde{\pi}$ be cusp forms in the space of π and $\tilde{\pi}$, and $f_s = \otimes_v f_s^{(v)} \in \mathscr{T}(s)$. In [5], the following formal identity was established;

Theorem. [5,§1]

$$\int_{G_n(k)\backslash G_n(\boldsymbol{A})} \int_{G_n(k)\backslash G_n(\boldsymbol{A})} E(i(g_1, g_2); f_s) \varphi(g_1) \tilde{\varphi}(g_2) dg_1 dg_2 = \prod_v \mathscr{Z}(f_s^{(v)}, \varphi_v, \tilde{\varphi}_v)$$

where

$$\mathscr{Z}(f_s^{(v)}, \varphi_v, \tilde{\varphi}_v) = \int_{G_n(k_v)} f_s^{(v)}(i(g_v, 1)) \langle \pi_v(g_v) \varphi_v, \tilde{\varphi_v} \rangle dg_v.$$

The product in the right hand side is taken over all the places v of k.

Our main result is the following;

Theorem. *The infinite product $L_{\mathrm{st}}(s, \pi)$ is analytically continued to a meromorphic function on whole of $\boldsymbol{C}$ and the number of poles of $L_{\mathrm{st}}(s, \pi)$ is finite.*

By the same method as [5,§6], it is possible to compute the local integrals at any unramified place v : let $\phi_s^{(v)}$ be the unique element of $\mathscr{T}_v(s)$, whose restriction to $K_{2n,v}$ is identically equal to 1, and let $\varphi_v^0 \in \pi_v$, $\tilde{\varphi}_v^0 \in \tilde{\pi}_v$ be non-zero $K_{n,v}$-fixed vectors. Then we have

$$d_v(s) \mathscr{Z}(\phi_s^{(v)}, \varphi_v^0, \tilde{\varphi}_v^0) = b_v(s) L_{\mathrm{st}}\left(s - n + \frac{1}{2}, \pi_v\right)$$

where $b(s)$ is defined by

$$b(s) = \prod_{l=1}^{n} \zeta_k(2s - 2l + 1) \zeta_k(2s - 2n - 2l + 2)$$

and $b_v(s)$ is the v-th local factor of $b(s)$.

To prove our main theorem, we need to study the local zeta integrals including ramified cases. The rest of this paper is devoted to this topic.

2. Local zeta integrals. Let us switch to the local notation. Write $F = k_v$ for a place v of k, and put $E = F \otimes_k \boldsymbol{K}$. Write $G_n = U_{n,n}(E/F)$, the F-points of G_n in §1. For simplicity we shall write $X = X(F)$ whenever X is a group defined over k in §1, and omit the sub-or superscript v in all cases.

We fix the embedding $G_n \times G_n \hookrightarrow G_{2n}$ (as in [5]) so that the subgroup P is of the form

$\left\{\begin{pmatrix} g & * \\ 0 & {}^t\bar{g}^{-1}\end{pmatrix} \mid g \in GL_{2n}(E)\right\}$. In particular, for

$g = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in G_n$,

$$(1)\ i(g, 1) = \begin{pmatrix} 1_n & 0 & 0 & 0 \\ C & D & 0 & -C \\ C & D-1_n & 1_n & -C \\ 1_n - A & -B & 0 & A \end{pmatrix}.$$

Let π be an irreducible admissible representation of G_n.

Proposition. *For $\varphi \in \pi$, $\tilde{\varphi} \in \tilde{\pi}$, and $f_s \in \mathcal{T}(s)$, the integral $\mathcal{Z}(f_s, \varphi, \tilde{\varphi})$ is absolutely convergent for $Re(s) >> 0$, and analytically continued to a rational function of q^{-s}, where q is the residue degree of F if F is non-archimedean, and to a product of gamma functions and a polynomial in s if F is archimedean. Moreover, if F is non-archimedean, it is possible to choose f_s, depending on φ, $\tilde{\varphi}$ so that $\mathcal{Z}(f_s, \varphi, \tilde{\varphi}) \equiv 1$.*

We shall reduce $\mathcal{Z}(f_s, \varphi, \tilde{\varphi})$ to integrals of the form

$$(2)\quad \int_{GL_m(E)}\int_{GL_m(E)} \Psi_1(z_1)\Psi_2(z_2) \mid \det(z_1) \mid_E^{s_1} \mid \det(z_2) \mid_E^{s_2} \omega(z_1^{-1}z_2)\, d^\times z_1 d^\times z_2$$

where Ψ_1, Ψ_2 are Schwartz-Bruhat functions on $\mathrm{Mat}_m(E)$, and ω is a certain matrix coefficient of an admissible representation of $GL_m(E)$. This can be written as a linear combination of the well-known Godement-Jacquet zeta integral ([3]); in fact, it is possible to take an elementary idempotent ξ of the maximal compact subgroup K_m of $GL_m(E)$ so that

$$\int_{K_m} \Psi_1(kz_1)\xi(k)\,dk = \Psi_1(z_1).$$

Then (2) is equal to

$$\int_{GL_m(E)}\int_{GL_m(E)} \Psi_1(z_1)\Psi_2(z_2) \mid \det(z_1) \mid^{s_1} \mid \det(z_2) \mid^{s_2} \left\{\int_{K_m} \xi(k)\omega(z_1^{-1}kz_2)\,dk\right\} d^\times z_1 d^\times z_2$$

and it is easy to see that

$$\int_{K_m} \xi(k)\omega(z_1^{-1}kz_2)\,dk = \sum_i \omega_i^{(1)}(z_1)\omega_i^{(2)}(z_2)$$

for suitable matrix coefficients $\omega_i^{(1)}$, $\omega_i^{(2)}$ of the same representation, analogous to the integral formula on spherical functions.

Now we begin the proof of the proposition. If E/F is split, i.e., if $E \simeq F \oplus F$, then $G_n \simeq GL_{2n}(F)$ and the study of the integral $\mathcal{Z}(f_s, \varphi, \tilde{\varphi})$ has been essentially done in [5,§6.1]; it can be written as (2), where $m = 2n$ and E is replaced by F, thus the proposition follows from [3].

Now let E/F be non-split, i.e., E/F is a quadratic field extension. So the possible archimedean case is $E = \boldsymbol{C}$, $F = \boldsymbol{R}$.

If the representation π is supercuspidal in non-archimedean case, then the matrix coefficient is compactly supported, for the center of the unitary group is compact, thus the claim is obvious. So, from now on, assume that the representation π is not supercuspidal.

By Jacquet's subrepresentation theorem, in non-archimedean case we may choose a maximal parabolic subgroup $P_n^{(r)}$ of G_n, whose Levi part is isomorphic to $GL_r(E) \times G_{n-r}$, and an admissible (resp. irreducible supercuspidal) representation τ (resp. τ') of $GL_r(E)$ (resp. G_{n-r}) such that π is a subrepresentation of the normalized induction $\mathrm{Ind}_{P_n^{(r)}}^{G_n}(\tau \otimes \tau')$. Then $\tilde{\pi}$, the contragradient of π, is realized as a quotient of $\mathrm{Ind}_{P_n^{(r)}}^{G_n}(\tilde{\tau} \otimes \tilde{\tau}')$. If $F = \boldsymbol{R}$, then π can be embedded into a principal series, and we may consider the case $r = n$ only, where we shall understand that τ' is trivial.

Consider $\varphi \in \pi$ as an element of $\mathrm{Ind}_{P_n^{(r)}}^{G_n}(\tau \otimes \tau')$, and regard $\tilde{\varphi} \in \tilde{\pi}$ as any of its representatives in $\mathrm{Ind}_{P_n^{(r)}}^{G_n}(\tilde{\tau} \otimes \tilde{\tau}')$. Then by rewriting the matrix coefficient of π we have

$$\mathcal{Z}(f_s, \varphi, \tilde{\varphi}) = \int_{G_n} f_s(i(g, 1))\langle\pi(g)\varphi, \tilde{\varphi}\rangle dg$$
$$= \int_{G_n} f_s(i(g, 1))\left(\int_{K_n} \langle\langle\varphi(gk), \tilde{\varphi}(k)\rangle\rangle dk\right)dg$$

and by the Iwasawa decomposition $G_n = U_n^{(r)} \times M_n^{(r)}K_n$, $dg = \delta_{P_n^{(r)}}(m)^{-1}\, dudmdk$,

$$= \int_{K_n\times K_n}\left\{\int_{U_n^{(r)}}\int_{M_n^{(r)}} f_s(i(umk_1k_2^{-1}, 1)) \langle\langle\varphi(umk_1), \tilde{\varphi}(k_2)\rangle\rangle \delta_{P_n^{(r)}}(m)^{-1}dudm\right\}dk_1dk_2.$$

Here, $\langle\langle\cdot\,, \cdot\rangle\rangle$ is the natural pairing of $\tau \otimes \tau'$ and $\tilde{\tau} \otimes \tilde{\tau}'$. The integration over $K_n \times K_n$ is not essential and we may only consider the case k_1, $k_2 = 1$ in the inner integral and the study of $\mathcal{Z}(f_s, \varphi, \tilde{\varphi})$ reduces to the integral

$$J := \int_{U_n^{(r)}}\int_{M_n^{(r)}} f_s(i(um, 1))\langle\langle\varphi(um), \tilde{\varphi}(1)\rangle\rangle \times \delta_{P_n^{(r)}}(m)^{-1}dudm$$
$$= \int_{U_n^{(r)}}\int_{M_n^{(r)}} f_s(i(um, 1))\langle\langle\tau(m)\cdot\varphi(1), \tilde{\varphi}(1)\rangle\rangle \times \delta_{P_n^{(r)}}(m)^{-1/2}dudm.$$

We may assume that $\varphi(1) = v \otimes w$, $\tilde{\varphi}(1) = \tilde{v} \otimes \tilde{w}$ where $v \in \tau$, $\tilde{v} \in \tilde{\tau}$, $w \in \tau'$, $\tilde{w} \in \tilde{\tau}'$. Put

$\omega_1(m_1) = \langle \tau(m_1)v, \tilde{v}\rangle_\tau, \quad \omega_2(m_2) = \langle \tau'(m_2)w, \tilde{w}\rangle_{\tau'}$
for $m_1 \in GL_r(E)$, $m_2 \in G_{n-r}$. Then our integral is of the form

$$J = \int_{GL_r(E)} \int_{G_{n-r}} \left\{ \int_{U_n^{(r)}} f_s(i(um_1m_2, 1))du \right\} \omega_1(m_1)\omega_2(m_2)\delta_{P_n^{(r)}}(m_1m_2)^{-1/2}dm_1dm_2.$$

As the function ω_2 is compactly supported (or trivial) on G_{n-r}, we may furthur reduce the problem to

$$(3) \qquad J' := \int_{GL_r(E)} \left\{ \int_{U_n^{(r)}} f_s(i(um_1, 1))du \right\} \omega_1(m_1)\delta_{P_n^{(r)}}(m_1)^{-1/2}dm_1.$$

Next, we write the integral over $U_n^{(r)}$ by the intertwining integral of principal series. By taking a suitable character μ_s of T_{2n}, $\mathcal{T}(s) = \mathrm{ind}_P^{G_{2n}}(\delta_s)$ can be made into a submodule of the principal series $\mathrm{Ind}_{B_{2n}}^{G_{2n}}(\mu_s)$. Explicitly μ_s is given by

$$\mu_s(\mathrm{diag}(t_1,\dots,t_{2n},\bar{t}_1^{-1},\dots,\bar{t}_{2n}^{-1})) = \prod_{i=1}^{2n} |t_i|_E^{s-2n+i-1/2}.$$

Lemma 1. *Let w_r be the element of W_{2n}, the relative Weyl group of G_{2n}, such that the action of w_r^{-1} on $X^*(A_{2n})$ is given by*

$$w_r^{-1} : \begin{cases} \chi_1 \mapsto \chi_1 & \chi_{n+1} \mapsto -\chi_{2r} \\ \vdots & \vdots \\ \chi_r \mapsto \chi_r & \chi_{n+r} \mapsto -\chi_{r+1} \\ \chi_{r+1} \mapsto \chi_{2r+1} & \chi_{n+r+1} \mapsto \chi_{n+r+1} \\ \vdots & \vdots \\ \chi_n \mapsto \chi_{n+1} & \chi_{2n} \mapsto \chi_{2n} \end{cases}$$

Then,

$$i(U_n^{(r)} \times 1) = \prod_{\alpha<0,\, w_r^{-1}\alpha>0} N_{2n,\alpha} = w_r\cdot\Big(\prod_{\alpha>0,\, w_r\alpha<0} N_{2n,\alpha}\Big)\cdot w_r^{-1}.$$

Proof. Put

$$\Phi^{-1}(r) := \{-(\chi_i - \chi_j) \mid n+1 \leqslant i \leqslant n+r < j \leqslant 2n\} \cup \{-2\chi_i \mid n+1 \leqslant i \leqslant n+r\} \cup \left\{ -(\chi_i + \chi_j) \,\middle|\, \begin{array}{l} r+1 \leqslant i \leqslant n < j \leqslant n+r, \text{ or } n+1 \leqslant i < j \leqslant n+r, \\ \text{or } n+1 \leqslant i \leqslant n+r < j \leqslant 2n \end{array} \right\}.$$

Then, by the explicit embedding (1), we have $i(U_n^{(r)} \times 1) = \Pi_{\alpha\in\Phi^-(r)} N_{2n,\alpha}$, and by an easy combinatorial discussion it is seen that $\Phi^-(r) = \{\alpha \in \Phi_{2n} \mid \alpha < 0,\ w_r^{-1}\alpha > 0\}$. □

Consider the case $f_s = \rho(h)\phi_s$, $h \in G_{2n}$, which can be regarded as an element of $\mathrm{Ind}_{B_{2n}}^{G_{2n}}(\mu_s)$. By the standard theory of intertwining operators, we have

$$(4) \qquad \int_{U_n^{(r)}} \phi_s(i(u,1)h)du = \int_{\prod_{\alpha>0, w_r\alpha<0} N_{2n,\alpha}} \phi_s(w_r n w_r^{-1}h)dn = c_{w_r}(\mu_s)\phi_{w_r^{-1}\cdot s}(w_r^{-1}h)$$

using the previous lemma. Here, $\phi_{w_r^{-1}\cdot s}$ is the unique element of $\mathrm{Ind}_{B_{2n}}^{G_{2n}}(w_r^{-1}\cdot\mu_s)$ such that $\phi_{w_r^{-1}\cdot s}|_{K_{2n}} \equiv 1$, and the factor $c_{w_r}(\mu_s)$ is the one given in [2].

Now we give a certain integral representation of $\phi_{w_r^{-1}\cdot s}$, using zeta functions of matrix rings. For $1 \leqslant k \leqslant 2n$, let Ψ_k (resp. $\Psi_k^{(0)}$) be a Schwartz-Bruhat function on $\mathrm{Mat}_{k\times 4n}(E)$ (resp. the characteristic function of $\mathrm{Mat}_{k\times 4n}(\mathcal{O}_E)$ in the non-archimedean case, and $\Psi_k^{(0)}(z) = \exp(-2\pi\mathrm{Tr}(z\,{}^t\bar{z}))$ in the case $E = \boldsymbol{C}$) and put for $h \in G_{2n}$,

$$F_{\Psi_k}(h;s) = \int_{GL_k(E)} \Psi_k((\overbrace{0,\dots,0}^{2n} \mid z, \underbrace{0,\dots,0}_{2n-k})\cdot h)\,|\det(z)|_E^s d^\times z$$

and $F_k(h;s) = F_{\Psi_k^{(0)}}(h;s)$. The right hand side converges absolutely for $Re(s) >> 0$, analytically continued to a meromorphic function on $\boldsymbol{C}$.

Lemma 2. *The function $F_{\Psi_r}(h;s_1)F_{\Psi_{2r}}(h;s_2)F_{\Psi_{n+r}}(h;s_3)F_{\Psi_{2n}}(h;s_4)$ belongs to $\mathrm{Ind}_{B_{2n}}^{G_{2n}}(w_r^{-1}\cdot\mu_s)$ and*

$$(5) \qquad \phi_{w_r^{-1}\cdot s}(h) = \eta(s)^{-1} \times F_r(h;s_1)F_{2r}(h;s_2)F_{n+r}(h;s_3)F_{2n}(h;s_4)$$

where $s_1 = 2s - 3n + 2r$, $s_2 = -s + 3n - r$, $s_3 = -r$, $s_4 = s$ *and*

$$\eta(s) = \prod_{k=1}^{r} \zeta_E(s_1 - k + 1) \prod_{k=1}^{2r} \zeta_E(s_2 - k + 1) \prod_{k=1}^{n+r} \zeta_E(s_3 - k + 1) \prod_{k=1}^{2n} \zeta_E(s_4 - k + 1).$$

Proof. For $t = \mathrm{diag}(t_1,\dots,t_{2n},\bar{t}_1^{-1},\dots,\bar{t}_{2n}^{-1}) \in T_{2n}$, we have, by a routine computation,

$$(6) \qquad w_r^{-1}\cdot\mu_s(t) = \prod_{i=1}^{r} |t_i|_E^{s-2n+i-1/2} \times \prod_{i=r+1}^{2r} |t_i|_E^{-s+n-2r+i-1/2} \prod_{i=2r+1}^{n+r} |t_i|_E^{s-2n+i-r-1/2} \times \prod_{i=n+r+1}^{2n} |t_i|_E^{s-2n+i-1/2}.$$

It is also easy to check that $F_{\Psi_k}(tnh;s) = \Pi_{i=1}^k |t_i|_E^s F_k(h;s)$ for t as above, $n \in N_{2n}$. Therefore,

$$(7) \qquad F_{\Psi_r}(tnh;s_1)F_{\Psi_{2r}}(tnh;s_2)F_{\Psi_{n+r}}(tnh;s_3)F_{\Psi_{2n}}(tnh;s_4) = \prod_{i=1}^{r} |t_i|_E^{s_1+s_2+s_3+s_4} \prod_{i=r+1}^{2r} |t_i|_E^{s_2+s_3+s_4} \times \prod_{i=2r+1}^{n+r} |t_i|_E^{s_3+s_4} \prod_{i=n+r+1}^{2n} |t_i|_E^{s_4} F_{\Psi_r}(h;s_1) \times F_{\Psi_{2r}}(h;s_2)F_{\Psi_{n+r}}(h;s_3)F_{\Psi_{2n}}(h;s_4).$$

Note that the normalizing factor of the principal series is given by

$$\delta_{B_{2n}}(t)^{1/2} = \prod_{i=1}^{2n} |t_i|_E^{2n-i+1/2}$$

Comparing (6) and (7), and

$$\begin{cases} s_1 + s_2 + s_3 + s_4 = s \\ s_2 + s_3 + s_4 = -s + 3n - 2r \\ s_3 + s_4 = s - r \\ s_4 = s \end{cases}$$

implies the first assertion. If $\Psi_k = \Psi_k^{(0)}$, it is of course right K_{2n}-invariant, thus normalizing each $F_k(h\,;s_i)$ so that is is equal to 1 at the identity, we have the lemma. □

We return to the integral (3). Let us choose a suitable representative for w_r, and let $m_1 = \iota(x, 1)x \in GL_r(E)$. By a direct matrix computation, we have

$$w_r^{-1} i(m_1, 1) w_r = \left(\begin{array}{cc|cc} x' & & & \\ & 1_{2(n-r)} & & \\ \hline & & {}^t\bar{x}'^{-1} & \\ & & & 1_{2(n-r)} \end{array}\right)$$

where $x' = \begin{pmatrix} 1_r & 0 \\ j_r - j_r x & j_r x j_r^{-1} \end{pmatrix} \in GL_{2r}(E)$.

Here j_r denotes the anti-diagonal $r \times r$ matrix whose non-zero entries are equal to 1. Applying (5), we have

$$\rho(h)\phi_{w_r^{-1}\cdot s}(w_r^{-1} i(m_1, 1)) = \rho(w_r^{-1}h)\phi_{w_r^{-1}\cdot s}(w_r^{-1} i(m_1, 1) w_r)$$

$$= \left(\prod_{k=1}^{r} \zeta_E(s_1 - k + 1)\right)^{-1}$$

$$\times \mid \det(x') \mid_E^{s_2+s_3+s_4} \int_{GL_r(E)} (\rho(w_r^{-1}h)\Psi_r^{(0)})$$

$$((\overbrace{0, \ldots, 0}^{2n} \mid (0, z) \cdot {}^t\bar{x}'^{-1}, \underbrace{0, \ldots, 0}_{2n-2r})) \mid \det(z) \mid_E^{s_1} d^\times z.$$

Finally, by (4),

$$J' = c_{w_r}(\mu_s) \int_{GL_r(E)} \rho(h)\phi_{w_r^{-1}\cdot s}(w_r^{-1} i(m_1, 1))\, \omega_1(m_1)$$

$$\times\, \delta_{P_n^{(r)}}(m_1)^{-1/2} dm_1 = c_{w_r}(\mu_s) \left(\prod_{k=1}^{r} \zeta_E(s_1 - k + 1)\right)^{-1}$$

$$\times \int_{GL_r(E)} \int_{GL_r(E)} \Psi_r'((z, 0) \cdot {}^t\bar{x}'^{-1})\, \omega_1(x)$$

$$\mid \det(z) \mid_E^{s_1} \mid \det(x) \mid_E^{s_2+s_3+s_4-n+r/2} d^\times z dx$$

for $f_s = \rho(h)\phi_s$. Here, Ψ_r' indicates the restriction of $\rho(w_r^{-1}h)\Psi_r^{(0)}$ to the $r \times 2r$-submatrices formed by the $(2n+1)$-st to the $(2n+2r)$-th columns, and used $\delta_{P_n^{(r)}}(m_1) = \mid \det(x) \mid_E^{2n-r}$. By a simple change of variable, this reduces to (2). Hence the assertions of the proposition, except for the last, are proved for $f_s = \rho(h)\phi_s$.

The remaining part of the proposition is seen as follows: in non-archimedean case, if $Re(s)$ is large enough, $\mathrm{Ind}_{B_{2n}}^{G_{2n}}(\mu_s)$ is generated by the vector ϕ_s as a G_{2n}-module ([2], 3.6), and therefore $\mathcal{T}(s)$ coincides with $\mathrm{Ind}_{B_{2n}}^{G_{2n}}(\mu_s)$. So for $Re(s)$ sufficiently large, it is enough to consider the case $f_s = \rho(h)\phi_s$. To prove the last part of the proposition, first choose the Schwartz-Bruhat functions Ψ_k so that the corresponding Godement-Jacquet zeta integrals are constant (this is possible by [3]). Yet in this case the element of $\mathrm{Ind}_{B_{2n}}^{G_{2n}}(w_r^{-1} \cdot \mu_s)$ given in Lemma 2. is an image of a finite sum of elements of the form $\rho(h)\phi_s$, by the intertwining integral. Therefore for $Re(s)$ large enough, it is possible to choose a finite number of $h_i \in G_{2n}$ such that $\mathcal{Z}(f_s, \varphi, \bar{\varphi}) \equiv$ const., where $f_s = \sum \rho(h_i)\phi_s$. By analytic continuation it is constant for all $s \in \mathbf{C}$. This completes the proof of the proposition.

References

[1] A. Borel: Automorphic L-functions. Proc. Symp. Pure Math., **33**, no. 2, 27–61 (1979).

[2] W. Casselman: The unramified principal series of $\mathfrak{p}$-adic groups I. The spherical function. Comp. Math., **40**, 387–406 (1980).

[3] R. Godement and H. Jacquet: Zeta functions of simple algebras. Lect. Notes in Math., vol. 260, Springer-Verlag, Berlin, Hidelberg, New York (1972).

[4] I. Piatetskii-Shapiro and S. Rallis: ε-factors of representations of classical groups. Proc. Nat. Acad. Sci. U.S.A., **83**, 4589–4593 (1986).

[5] I. Piatetskii-Shapiro and S. Rallis: L-functions for classical groups. Lect. Notes in Math., vol. 1254, Springer-Verlag, Berlin, Hidelberg, New York (1987).