Extensions of Hölder-McCarthy and Kantorovich Inequalities and Their Applications*)

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Abstract: Extensions of Hölder-McCarthy and Kantorovich inequalities are given and their applications to the order preserving power inequalities are also given.

§1. Extensions of Hölder-McCarthy and Kantorovich inequalities. This paper is an early announcement of [3], [4], and [5]. An operator means a bounded linear operator on a Hilbert space H. The celebrated Kantorovich inequality asserts that if A is positive operator on H such that $M \geq A \geq m > 0$, then $(A^{-1}x, x)(Ax, x) \leq \frac{(m+M)^2}{4mM}$ holds for every unit vector x in H. At first we state extensions of Kantorovich inequality.

Multiple positive definite operator case.

Theorem 1.1 [4]. *Let A_j be positive operator on a Hilbert space H satisfying $MI \geq A_j \geq mI (j = 1,2, \ldots, k)$, where $M > m > 0$. Let $f(t)$ be a real valued continuous convex function on $[m, M]$ and also let $x_1, x_2, \ldots, x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k} \| x_j \|^2 = 1$. Then the following inequality holds;*

$$\sum_{j=1}^{k} (f(A_j)x_j, x_j) \leq \frac{(mf(M) - Mf(m))}{(q-1)(M-m)} \left(\frac{(q-1)(f(M) - f(m))}{q(mf(M) - Mf(m))} \right)^q \left(\sum_{j=1}^{k} (A_j x_j, x_j) \right)^q$$

under any one of the following conditions (i) *and* (ii) *respectively;*

(i) $f(M) > f(m)$, $\frac{f(M)}{M} > \frac{f(m)}{m}$ *and*

$$\frac{f(m)}{m} q \leq \frac{f(M) - f(m)}{M - m} \leq \frac{f(M)}{M} q$$

holds for any real number $q > 1$,

(ii) $f(M) < f(m)$, $\frac{f(M)}{M} < \frac{f(m)}{m}$ *and*

$$\frac{f(m)}{m} q \leq \frac{f(M) - f(m)}{M - m} \leq \frac{f(M)}{M} q$$

holds for any real number $q < 0$.

Corollary 1.2 [4]. *Let A_j be positive operator on a Hilbert space H satisfying $MI \geq A_j \geq mI (j = 1,2, \ldots, k)$, where $M > m > 0$. Let $x_1, x_2, \ldots, x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k} \| x_j \|^2 = 1$. Then the following inequality holds;*

$$\sum_{j=1}^{k} (A_j^p x_j, x_j) \leq \frac{(mM^p - Mm^p)}{(q-1)(M-m)} \left(\frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)} \right)^q \left(\sum_{j=1}^{k} (A_j x_j, x_j) \right)^q$$

under any one of the following conditions (i) *and* (ii) *respectively;*

(i) $m^{p-1} q \leq \frac{M^p - m^p}{M - m} \leq M^{p-1} q$ *holds for any real numbers $p > 1$ and $q > 1$,*

(ii) $m^{p-1} q \leq \frac{M^p - m^p}{M - m} \leq M^{p-1} q$ *holds for any real numbers $p < 0$ and $q < 0$.*

Corollary 1.2 becomes the following Corollary 1.3 if we put $q = p$.

Corollary 1.3 [4]. *Let A_j be positive operator on a Hilbert space H satisfying $MI \geq A_j \geq mI (j = 1,2, \ldots, k)$, where $M > m > 0$. Let $x_1, x_2, \ldots, x_k$ be any finite number of vectors in H such that $\sum_{j=1}^{k} \| x_j \|^2 = 1$. Then the following inequality holds for any real number $p \notin [0, 1]$;*

$$\sum_{j=1}^{k} (A_j^p x_j, x_j) \leq \frac{(mM^p - Mm^p)}{(p-1)(M-m)} \left(\frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)} \right)^p \left(\sum_{j=1}^{k} (A_j x_j, x_j) \right)^p.$$

Corollary 1.3 can be considered as an extension of the following Theorem A by Ky Fan.

Theorem A [1] (Ky Fan). *Let A be a positive definite Hermitian matrix of order n with all its eigenvalues contained in the closed interval $[m, M]$, where $M > m > 0$. Let $x_1, x_2, \ldots, x_k$ be any finite*

*) Dedicated to Professor Shigeru Kita on his 88th birthday with respect and affection.

number of vectors in the unitary n-space such that $\sum_{j=1}^{k} \| x_j \|^2 = 1$. *Then for every integer $p \neq 0,1$ (not necessary positive) we have;*

$$\sum_{j=1}^{k} (A^p x_j, x_j) \leq \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(mM^p - Mm^p)^{p-1}(M-m)} \left(\sum_{j=1}^{k} (Ax_j, x_j)\right)^p.$$

In particular $\left(\sum_{j=1}^{k} (Ax_j, x_j)\right)\left(\sum_{j=1}^{k} (A^{-1}x_j, x_j)\right) \leq \frac{(m+M)^2}{4mM}$.

Corollary 1.4 [3]. *Let A be positive operator on a Hilbert space H satisfying $MI \geq A \geq mI$, where $M > m > 0$. Then the following inequalities hold for every unit vector x in H.*

(i) $(Ax, x)^p (A^{-1}x, x) \leq \frac{p^p}{(p+1)^{p+1}} \frac{(m+M)^{p+1}}{mM}$

for any p such that $\frac{m}{M} \leq p \leq \frac{M}{m}$

(ii) $(A^2 x, x) \leq \frac{p^p}{(p+1)^{p+1}} \frac{(m+M)^{p+1}}{(mM)^p} (Ax, x)^{p+1}$

for any p such that $\frac{m}{M} \leq p \leq \frac{M}{m}$.

(i) in Corollary 1.4 with $p = 1$ becomes the Kantorovich inequality.

Multiple positive definite matrix case.

Theorem 1.5 [4]. *Let A_j be positive definite Hermite matrices of order n with eigenvalues contained in the interval $[m, M]$, where $M > m > 0$. Let $f(t)$ be a real valued continuous convex function on $[m, M]$ and also $U_j (j = 1,2, \ldots, k)$ are $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$. Then the following inequality holds;*

$$\sum_{j=1}^{k} U_j f(A_j) U_j^* \leq \frac{(mf(M) - Mf(m))}{(q-1)(M-m)} \left(\frac{(q-1)(f(M) - f(m))}{q(mf(M) - Mf(m))}\right)^q \left(\sum_{j=1}^{k} U_j A_j U^*{}_j\right)^q$$

holds under any one of the following conditions (i) *an* (ii);

(i) $f(M) > f(m)$, $\frac{f(M)}{M} > \frac{f(m)}{m}$ *and*

$$\frac{f(m)}{m} q \leq \frac{f(M) - f(m)}{M - m} \leq \frac{f(M)}{M} q$$

holds for any real number $q > 1$,

(ii) $f(M) < f(m)$, $\frac{f(M)}{M} < \frac{f(m)}{m}$ *and*

$$\frac{f(m)}{m} q \leq \frac{f(M) - f(m)}{M - m} \leq \frac{f(M)}{M} q$$

holds for any real number $q < 0$.

Corollary 1.6 [4]. *Let A_j be positive definite Hermite matrices of order n with eigenvalues contained in the interval $[m, M]$, there $M > m > 0$. If $U_j (j = 1,2, \ldots, k)$ are $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$. Then the following inequality holds;*

$$\sum_{j=1}^{k} U_j A_j^p U_j^* \leq \frac{(mM^p - Mm^p)}{(q-1)(M-m)} \left(\frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)}\right)^q \left(\sum_{j=1}^{k} U_j A_j U^*{}_j\right)^q$$

holds under any one of the following conditions (i) and (ii);

(i) $m^{p-1} q \leq \frac{f(M) - f(m)}{M - m} \leq M^{p-1} q$ *holds for any real numbers $p > 1$ and $q > 1$,*

(ii) $m^{p-1} q \leq \frac{f(M) - f(m)}{M - m} \leq M^{p-1} q$ *holds for any real numbers $p < 0$ and $q < 0$.*

If we put $q = p$ in Corollary 1.6, we have the following result which is a matrix version of Theorem A by Ky Fan.

Corollary 1.7 [4]. *Let A_j be positive definite Hermite matrices of order n with eigenvalues contained in the interval $[m, M]$, where $M > m > 0$. Also let $U_j (j = 1,2, \ldots, k)$ be $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$. Then for any real number p such that $p \notin [0,1]$, the following inequality holds;*

$$\sum_{j=1}^{k} U_j A_j^p U_j^* \leq \frac{(mM^p - Mm^p)}{(p-1)(M-m)} \left(\frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)}\right)^p \left(\sum_{j=1}^{k} U_j A_j U^*{}_j\right)^p.$$

Corollary 1.8 [3]. *Let $A_j (j = 1, \ldots, k)$ be positive definite Hermite matrices of order n, with eigenvalues contained in the interval $[m, M]$, where $M > m > 0$. Also let $U_j (j = 1, \ldots, k)$ be $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$. Then the following inequalities hold;*

(i) $$\sum_{j=1}^{k} U_j A_j^{-1} U_j^* \leq \frac{p^p}{(p+1)^{p+1}} \frac{(m+M)^{p+1}}{mM} \left(\sum_{j=1}^{k} U_j A_j U_j^*\right)^{-p}$$

for any positive p such that $\frac{m}{M} \leq p \leq \frac{M}{m}$.

(ii) $$\sum_{j=1}^{k} U_j A_j^2 U_j^* \leq \frac{p^p}{(p+1)^{p+1}} \frac{(m+M)^{p+1}}{(mM)^p} \left(\sum_{j=1}^{k} U_j A_j U_j^*\right)^{p+1}$$

for any positive p such that $\frac{m}{M} \leq p \leq \frac{M}{m}$.

Corollary 1.8 with $p = 1$ becomes the following Theorem B.

Theorem B [7]. *Let A_j $(j = 1, \ldots, k)$ be positive definite Hermite matrices of order n, with eigenvalues contained in the interval $[m, M]$, where $M > m > 0$. If U_j $(j = 1, \ldots, k)$ are $r \times n$ matrices such that $\sum_{j=1}^{k} U_j U_j^* = I$, then the following inequalities hold;*

(i) $$\sum_{j=1}^{k} U_j A_j^{-1} U_j^* \leq \frac{(m+M)^2}{4mM} \left(\sum_{j=1}^{k} U_j A_j U_j^*\right)^{-1}$$

(ii) $$\sum_{j=1}^{k} U_j A_j^2 U_j^* \leq \frac{(m+M)^2}{4mM} \left(\sum_{j=1}^{k} U_j A_j U_j^*\right)^{2}.$$

Next we state results on complementary inequality of Hölder-McCarthy inequality.

Theorem 1.9 [5]. *Let A be positive operators on a Hilbert space H satisfying $M \geq A \geq m > 0$. Then the following inequality holds for every unit vector x*

(i) *In case $p > 1$:*

$$(Ax, x)^p \leq (A^p x, x) \leq K_+(m, M)(Ax, x)^p$$

where $K_+(m, M)$

$$= \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}.$$

(ii) *In case $p < 0$:*

$$(Ax, x)^p \leq (A^p x, x) \leq K_-(m, M)(Ax, x)^p$$

where $K_-(m, M)$

$$= \frac{(mM^p - Mm^p)}{(p-1)(M-m)} \left(\frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)}\right)^p.$$

Recently the following interesting complementary inequality of Hölder-McCarthy inequality [6] is shown in [2].

Theorem B ([2]). *Let A and B be positive operators on a Hilbert space H satisfying $M_1 \geq A \geq m_1 > 0$ and $M_2 \geq B \geq m_2 > 0$. Let p and q be $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequality holds for every vector x*

$$(B^q \#_{1/p} A^p x, x) \leq (A^p x, x)^{1/p} (B^q x, x)^{1/q}$$
$$\leq \lambda\left(p, \frac{m_1}{M_2^{q-1}}, \frac{M_1}{m_2^{q-1}}\right)^{1/p} (B^q \#_{1/p} A^p x, x),$$

where $\lambda(p, m, M)$

$$= \left\{\frac{1}{p^{1/p} q^{1/q}} \frac{M^p - m^p}{(M-m)^{1/p} (mM^p - Mm^p)^{1/q}}\right\}^p.$$

We give the following extension of Theorem B by considering the case $p < 0$ and $1 > q > 0$.

Theorem 1.10 [5]. *Let A and B be positive operators on a Hilbert space H satisfying $M_1 \geq A \geq m_1 > 0$ and $M_2 \geq B \geq m_2 > 0$. Let p and q be conjugate real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequalities hold for every vector x and real numbers r and s:*

(i) *In case $p > 1$, $q > 1$, $r \geq 0$ and $s \geq 0$:*

(1.8) $$(B^r \#_{1/p} A^s x, x) \leq (A^s x, x)^{1/p} (B^r x, x)^{1/q}$$
$$\leq K_+\left(\frac{m_1^{s/p}}{M_2^{r/p}}, \frac{M_1^{s/p}}{m_2^{r/p}}\right)^{1/p} (B^r \#_{1/p} A^s x, x).$$

(ii) *In case $p < 0$, $1 > q > 0$, $r \geq 0$ and $s \leq 0$:*

(1.9) $$(B^r \#_{1/p} A^s x, x) \geq (A^s x, x)^{1/p} (B^r x, x)^{1/q}$$
$$\geq K_-\left(\frac{m_1^{s/p}}{m_2^{r/p}}, \frac{M_1^{s/p}}{M_2^{r/p}}\right)^{1/p} (B^r \#_{1/p} A^s x, x).$$

where $K_+(,)$ and $K_-(,)$ are the same as defined in Theorem 1.3. In particular,

(i) *In case $p > 1$ and $q > 1$,:*

(1.10) $$(B^q \#_{1/p} A^p x, x) \leq (A^p x, x)^{1/p} (B^q x, x)^{1/q}$$
$$\leq K_+\left(\frac{m_1}{M_2^{q-1}}, \frac{M_1}{m_2^{q-1}}\right)^{1/p} (B^q \#_{1/p} A^p x, x).$$

(ii) *In case $p < 0$ and $1 > q > 0$:*

(1.11) $$(B^q \#_{1/p} A^p x, x) \geq (A^p x, x)^{1/p} (B^q x, x)^{1/q}$$
$$\geq K_-\left(\frac{m_1}{M_2^{q-1}}, \frac{M_1}{M_2^{q-1}}\right)^{1/p} (B^q \#_{1/p} A^p x, x).$$

Remark 1.1. We remark that (1.10) in Theorem 1.10 just equals to Theorem B and (1.10) is equivalent to (1.8) and also (1.11) is equivalent to (1.9).

§2. Applications of Theorem 1.9 to order preserving power inequalities.

$0 < A \leq B$ ensures $A^p \leq B^p$ for any $p \in [0,1]$ by well known Löwner-Heinz theorem. However it is well known that $0 < A \leq B$ does not always ensure $A^p \leq B^p$ for any $p > 1$. Related to this result, a simple proof of the following interesting result is given in [2].

Theorem C [2]. *Let $0 < A \leq B$ and $0 < m \leq A \leq M$. Then*

$$A^p \leq \left(\frac{M}{m}\right)^p B^p \quad \textit{for } p \geq 1.$$

We obtained the following result related to Theorem C.

Theorem 2.1 [5]. *Let A and B be positive operators on a Hilbert space H such that $M_1 \geq A \geq m_1 > 0$, $M_2 \geq B \geq m_2 > 0$ and $0 < A \leq B$. Then*

(1-A) $$A^p \leq K_{1,p} B^p \leq \left(\frac{M_1}{m_1}\right)^{p-1} B^p$$

and

(2-B) $$A^p \leq K_{2,p} B^p \leq \left(\frac{M_2}{m_2}\right)^{p-1} B^p$$

holds for any $p \geq 1$, where $K_{1,p}$ and $K_{2,p}$ are defined by the following

(2.1) $$K_{1,p} = \frac{(p-1)^{p-1}}{p^p (M_1 - m_1)} \frac{(M_1^p - m_1^p)^p}{(m_1 M_1^p - M_1 m_1^p)^{p-1}}$$

and

(2.2) $$K_{2,p} = \frac{(p-1)^{p-1}}{p^p (M_2 - m_2)} \frac{(M_2^p - m_2^p)^p}{(m_2 M_2^p - M_2 m_2^p)^{p-1}}.$$

Remark 2.1. (1-A) and (2-B) of Theorem 2.1 are more precise estimation than Theorem C since $K_{j,p} \leq \left(\frac{M_j}{m_j}\right)^{p-1} \leq \left(\frac{M_j}{m_j}\right)^p$ holds for $j = 1,2$ and $p \geq 1$ ([5]).

Results in [3], [4], and [5] will appear elsewhere and other results related to this paper are discussed in [3], [4], and [5].

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