On Ranks of the Stable Derivation Algebra and Deligne's Problem

By Hiroshi TSUNOGAI

Department of Mathematics, Sophia University

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1. Introduction. Among the outer Galois representations attached to algebraic varieties, the one attached to $X = \boldsymbol{P}^1 \setminus \{0,1,\infty\}$ is most fundamental. In this article, we consider the Lie algebraization of the pro-l outer Galois representation. The descending central filtration of the fundamental group $\pi_1(\bar{X})$ induces a central filtration into the absolute Galois group $G_{\boldsymbol{Q}} = \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$, and the associating graded module naturally turns into a Lie algebra, which we call the *Galois Lie algebra* in this article. In [6], Ihara introduced the *stable derivation algebra,* which includes the Galois Lie algebra. But we do not know its presentation by generators and relations, nor any explicit formulae for the ranks of its graded components. On the other hand, relating to the philosophy that not only cohomology groups but also fundamental groups are motivic [2], Deligne proposed a conjecture on the structure of the Galois Lie algebra, which is sometimes called *Deligne's motivic conjecture.* We report here that this conjecture is valid in degrees less than 13. We also obtain a partial evidence for the conjecture in degrees less than 18.

2. The Galois Lie algebra. Let l be a prime number and consider the pro-l *outer Galois representation* $\varphi_X^{(l)}$ attached to $X = \boldsymbol{P}^1 \setminus \{0,1,\infty\}$

(2.1) $$\varphi_X^{(l)} : G_{\boldsymbol{Q}} \to \mathrm{Out}\pi_1^{(l)}(\bar{X}),$$

where $G_{\boldsymbol{Q}}$ denote the absolute Galois group $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ of $\boldsymbol{Q}$, and $\pi_1^{(l)}(\bar{X})$ is the pro-l fundamental group of $\bar{X} = X \times \bar{\boldsymbol{Q}}$, which is isomorphic to the free pro-l group of rank two. By introducing the descending central filtration into $\pi_1^{(l)}(\bar{X})$, we have an injective homomorphism $\varphi_{\mathscr{G}}$ of graded Lie algebras

(2.2) $$\varphi_{\mathscr{G}} : \mathscr{G} \to \mathrm{Out}\mathscr{F}_2,$$

where $\mathrm{Out}\mathscr{F}_2$ denotes the outer derivation algebra of the free Lie algebra $\mathscr{F}_2$ of rank two [5, 8, and 9]. We call $\varphi_{\mathscr{G}}$ the *Lie algebraization* of φ_X. We also have a lifting of $\varphi_{\mathscr{G}}$ into $\mathrm{Der}\mathscr{F}_2$ and the image of $\mathscr{G}$ is included in a subalgebra $\mathscr{D}_5$ of $\mathrm{Der}\mathscr{F}_2$, whose definition we shall state later. For odd m greater than or equal to 3, the component $\mathrm{gr}^m\mathscr{G}$ of $\mathscr{G}$ of degree m has a non-trivial element σ_m which does not belong to $[\mathscr{G}, \mathscr{G}]$. This fact is deduced from the explicit formula of Ihara's power series representation and the non-trivialitity of Soulé's character. These σ_m's are called *Soulé elements.* Note that for m as above, σ_m is unique up to scalar multiple and modulo $[\mathscr{G}, \mathscr{G}]$.

3. Deligne's problem. Let $\mathscr{F}$ be the free graded Lie algebra over $\boldsymbol{Q}$ generated by the symbols $\tau_m (m : \mathrm{odd} \geq 3)$ of degree m. We have a homomorphism $\psi : \mathscr{F} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \to \mathscr{G} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$ which maps τ_m to σ_m. Deligne proposed the following conjecture:

Conjecture (Deligne). *The homomorphism $\psi : \mathscr{F} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \to \mathscr{G} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$ would be an isomorphism between graded Lie algebras.*

We obtained an affirmative answer for this conjecture in low degrees by a computational method.

Theorem 4. 1. *The conjecture is valid in degrees less than* 13. *Namely, the homomorphism ψ gives an isomorphism*

$$\Big(\mathscr{F} / \bigoplus_{m \geq 13} \mathrm{gr}^m \mathscr{F}\Big) \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \simeq \Big(\mathscr{G} / \bigoplus_{m \geq 13} \mathrm{gr}^m \mathscr{G}\Big) \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$$

between graded Lie algebras.

2. (*a partial result*) *For $m \leq 15$ and $m = 17$, the homomorphisms*

$$\mathrm{gr}^m \psi : \mathrm{gr}^m \mathscr{F} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \to \mathrm{gr}^m \mathscr{G} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$$

between the m-th degree components are injective.

We shall show the theorem by relating the Galois Lie algebra to a more computable object, the *stable derivation algebra,* and by explicit computation of derivations using computers.

Remark 5. In [4 and 5], Ihara studied the structure of these graded Lie algebras, and

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obtained the non-vanishing of Lie brackets of two Soulé elements and the table of the ranks of $\mathrm{gr}^m\mathscr{G}$ for $m \leq 8$. Extending his result, Matsumoto [8] obtained the non-vanishing of iterated Lie brackets of Soulé elements and showed that the conjecture is valid in degrees less than 11 and that $\mathrm{gr}^m\phi$ is injective for $m \leq 12$. Combining his result and the result of Ihara-Takao on linear independence of simple Lie brackets [7], one can obtain the injectivity of $\mathrm{gr}^{14}\phi$ by the argument based on the bi-degree filtration (*loc. cit.* and [9]). Our theorem claims, in addition to the known results mentioned above, the bijectivity of $\mathrm{gr}^m\phi$ for $m = 11$, 12 and the injectivity of $\mathrm{gr}^m\phi$ for $m =$ 13, 15, 17, whose proof depends on the determination of the dimensions of $\mathrm{gr}^m\mathscr{D}_5$ for $m = 11$, 12 and the whole coordinate of σ_{11}.

6. The stable derivation algebra. Let $\mathscr{F}_2$ be the free graded Lie algebra over $\boldsymbol{Q}$ generated by x, y, z with $x + y + z = 0$ whose degrees are one. The *stable derivation algebra* $\mathscr{D}_5$ is the subalgebra of the derivation algebra $\mathrm{Der}\mathscr{F}_2$ of $\mathscr{F}_2$ defined to be

$$\text{(6.1)} \quad \mathscr{D}_5 = \left\{ D \in \mathrm{Der}\mathscr{F}_2 \;\middle|\; \begin{array}{l} D(x) = 0, D(y) = [y, f]\,(\exists f = f(x, y) \in \mathscr{F}_2) \\ \quad \text{s.t. (I), (II) and (III), (V)} \end{array} \right\},$$

(I) (the degree one component of f) $= 0$,

(II) $f(x, y) + f(y, x) = 0$,

(III) $[y, f(x, y)] + [z, f(x, z)] = 0$,

(V) $f(x_{12}, x_{23}) + f(x_{34}, x_{45}) + f(x_{51}, x_{12}) + f(x_{23}, x_{34}) + f(x_{45}, x_{51}) = 0$ in $\mathscr{P}_5$.

(For the precise definition of the Lie algebra $\mathscr{P}_5$ and its special elements x_{ij}'s, see Ihara [6].) We also consider another Lie algebra $\mathscr{D}_4$, which is defined to be the one consisting of the derivations satisfying (I), (II), and (III). For a derivation $D \in \mathscr{D}_4$, we call $f \in \mathscr{F}_2$ in the above definition (which is unique for D) the *coordinate* of D.

7. The method of computation. For each m, we have homomorphisms

$$\text{(7.1)} \quad \mathrm{gr}^m\mathscr{F} \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \xrightarrow{\mathrm{gr}^m\phi} \mathrm{gr}^m\mathscr{G} \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l \xrightarrow{\mathrm{gr}^m\varphi_{\mathscr{G}}} \mathrm{gr}^m\mathscr{D}_5 \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l \subset \mathrm{gr}^m\mathscr{D}_4 \otimes_{\boldsymbol{Q}} \boldsymbol{Q}_l,$$

where $\mathrm{gr}^m\varphi_{\mathscr{G}}$ is injective. We can show that $\mathrm{gr}^m\phi$ is an isomorphism if $\mathrm{gr}^m\phi$ is injective and if $\dim \mathrm{gr}^m\mathscr{F} = \dim \mathrm{gr}^m\mathscr{D}_5$.

For the dimension of $\mathrm{gr}^m\mathscr{F}$, we have an explicit formula:

$$\text{(7.2)} \quad \mathrm{rank}\,\mathrm{gr}^m\mathscr{F} = \sum_{d|m} \frac{\mu(d)}{d} \sum_{\frac{m}{3d} \leq n \leq \frac{m-d}{2d}} \frac{1}{n} \binom{n}{\frac{m}{d} - 2n},$$

which can be shown by a similar method as in the proof of Witt formula [10]. The explicit formulae for the dimension of $\mathrm{gr}^m\mathscr{D}_4$ have been obtained by Ihara and Deligne independently [4]. Because no explicit formulae have been found for the dimension of $\mathrm{gr}^m\mathscr{D}_5$, we must determine the dimension of the subspace of $\mathrm{gr}^m\mathscr{F}_2$ satisfying (II), (III), and (V) to obtain it by means of large computation with computers except for some small m. On the other hand, to show the injectivity of $\mathrm{gr}^m\phi$, we must show the linear independence of Hall monomials [1,3,8, and 9] constructed from σ_m's, which are listed in the table for lower degrees. To check this in a definite degree computationally, we need to know the coordinates $f = f_m$ of σ_m in $\mathrm{gr}^m\mathscr{D}_5$. Hence we take the following strategy:

1. First determine the subspace of $\mathrm{gr}^m\mathscr{F}_2$ satisfying (II), (III), and (V), and also determine the whole coordinate of σ_m if m is odd.
2. Using the coordinate of σ_m's, determine the coordinates of Hall monomials made of them by actural computation, and check the linear independence among them.

For the computation of the coordinates of Lie brackets, note that if we let f_i the coordinates of D_i $(i = 1,2)$, the coordinate of $[D_1, D_2]$ is $D_1(f_2) - D_2(f_1) + [f_1, f_2]$.

For the proof of the theorem, we need the following computational result:

Lemma 8. *The component of degree* 11 *of* $\mathscr{D}_5$ *is of rank* 2. *The component of degree* 12 *of* $\mathscr{D}_5$ *is of rank* 2.

Since $\mathrm{gr}^{11}\mathscr{G}$ includes two linearly independent elements σ_{11} and $[\sigma_3, [\sigma_3, \sigma_5]]$ and $\mathrm{gr}^{12}\mathscr{G}$ includes two linearly independent elements $[\sigma_3, \sigma_9]$ and $[\sigma_5, \sigma_7]$, the ranks of $\mathrm{gr}^{11}\mathscr{G}$ and of $\mathrm{gr}^{12}\mathscr{G}$ are determined to be exactly two. This computation also gives us the whole coordinate of σ_{11} (up to scalar multiple and modulo $[\sigma_3, [\sigma_3, \sigma_5]]$).

We also check injectivity of $\mathrm{gr}^m\phi$ for some m by carrying out the computation explicitly using the coordinates of Soulé elements $\sigma_m (m = 3, 5, 7, 9, 11)$. We found that for $m \leq 15$ or $m = 17$, $\mathrm{gr}^m\phi$ is injective. To show the injectivity of $\mathrm{gr}^{16}\phi$, we need more information about the coordinate of σ_{13}. We found, however, the linear independence of the other four monomials listed in

Table. Ranks of graded Lie algebras

m	$\mathrm{gr}^m\mathscr{D}_4$	$\mathrm{gr}^m\mathscr{D}_5$	$\mathrm{gr}^m\mathscr{F}$	$\mathrm{gr}^m\mathscr{G}$	Linearly independent elements in $\mathrm{gr}^m\mathscr{G}$
1	0	0	0	0	
2	0	0	0	0	
3	1	1	1	1	σ_3
4	0	0	0	0	
5	1	1	1	1	σ_5
6	0	0	0	0	
7	2	1	1	1	σ_7
8	1	1	1	1	$[\sigma_3, \sigma_5]$
9	4	1	1	1	σ_9
10	2	1	1	1	$[\sigma_3, \sigma_7]$
11	9	2	2	2	σ_{11}, $[\sigma_3, [\sigma_3, \sigma_5]]$
12	7	2	2	2	$[\sigma_3, \sigma_9]$, $[\sigma_5, \sigma_7]$
13	21		3	≥ 3	σ_{13}, $[\sigma_3, [\sigma_3, \sigma_7]]$, $[\sigma_5, [\sigma_3, \sigma_5]]$
14	24		3	≥ 3	$[\sigma_3, \sigma_{11}]$, $[\sigma_5, \sigma_9]$, $[\sigma_3, [\sigma_3, [\sigma_3, \sigma_5]]]$
15	56		4	≥ 4	σ_{15}, $[\sigma_3, [\sigma_3, \sigma_9]]$, $[\sigma_5, [\sigma_3, \sigma_7]]$, $[\sigma_7, [\sigma_3, \sigma_5]]$
16	75		5	≥ 4	$\{$ $([\sigma_3, \sigma_{13}],)$ $[\sigma_5,\sigma_{11}]$, $[\sigma_7, \sigma_9]$, $[\sigma_3,[\sigma_3, [\sigma_3, \sigma_7]]]$, $[\sigma_5, [\sigma_3, [\sigma_3, \sigma_5]]]$
17	163		7	≥ 7	$\{$ σ_{17}, $[\sigma_3, [\sigma_3, \sigma_{11}]]$, $[\sigma_5, [\sigma_3, \sigma_9]]$, $[\sigma_5, [\sigma_5, \sigma_7]]$, $[\sigma_7, [\sigma_3, \sigma_7]]$, $[\sigma_9, [\sigma_3, \sigma_5]]$, $[\sigma_3, [\sigma_3, [\sigma_3, [\sigma_3, \sigma_5]]]]$

the table.

Remark 9. For computer calculation, the author made a kit of routines executing the Hall basis algorithm. Except some parts specified for this computation, this program is available electronically at ftp://ftp. math. metro-u. ac. jp/tnt/lie-1.0.tar.gz.

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