Remarks on the Periodic Solution of the Heat Convection Equation in a Perturbed Annulus Domain

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1. Introduction. We consider the heat convection equation in a time-dependent bounded domain $\Omega(t)$ of $\boldsymbol{R}^2$ which varies periodically with period T_p.

$$(1)\quad \begin{cases} u_t + (u\cdot\nabla)u = -(\nabla p)/\rho + \\ \qquad \{1-\alpha(\theta - T_0)\}g + \nu\Delta u & \text{in } \hat{\Omega}, \\ \qquad \operatorname{div} u = 0 & \text{in } \hat{\Omega}, \\ \theta_t + (u\cdot\nabla)\theta = \kappa\Delta\theta & \text{in } \hat{\Omega}, \end{cases}$$

$$(2)\quad u|_{\partial\Omega(t)} = \beta(x, t),\ \theta|_{\Gamma_0} = T_0 > 0,\ \theta|_{\Gamma(t)} = 0 \quad \text{for any } t\in(0,\infty),$$

$$(3)\quad u(\cdot, t+T_p) = u(\cdot, t),\ \theta(\cdot, t+T_p) = \theta(t), \quad \text{in } \Omega(0),$$

where $\hat{\Omega} = \cup_{0<t<\infty}\Omega(t)\times\{t\}$ and $\partial\Omega(t)$ (the boundary of $\Omega(t)$) consists of two smooth components, i.e. $\partial\Omega(t) = \Gamma_0 \cup \Gamma(t)$, and Γ_0 is the inner boundary which bounds a compact set K, while the outer boundary $\Gamma(t)$ is a smooth one with respect to both x and t. We assume that the set K includes the origine O and $\Omega(t)$ is included in a ball $B_d = B(O, d/2)$. We put $B = B_d\setminus K$. Moreover, $u = u(x, t)$ is the velocity vector, $p = p(x, t)$ is the pressure and $\theta = \theta(x, t)$ is the temperature; ν, κ, α, ρ and $g = g(x)$ are the kinematic viscosity, the thermal conductivity, the coefficient of volume expansion, the density at $\theta = T_0$ and the gravitational vector, respectively. (Hereafter, we denote the heat convection equation by HC equation).

As for the 3-dimensional problems, we proved the existence, uniqueness and the stability of the periodic strong solutions in [9] and [10] when the data are small, while Morimoto [8] obtained the periodic weak solutions. Recently, Inoue-Ôtani [6] studied and got the periodic strong solution under their small type conditions when the space dimention $n = 2$ or 3 (in time-dependent domains). On the other hand , for the 2-dimensional cases, we obtained, in [14], a sufficient condition for the existence of the periodic strong solution in the form of a certain relation between given data including a time period, but not including the magnitude of b which is an extension of the boundary function $\beta(x, t)$. The purpose of the present paper is to improve the result of our previous one [14] and to remove the small type condition on the boundary data of the fluid velocity. (We announced the results of this paper in [15]).

2. Preliminaries. First, we make assumptions:

(A1) For each fixed $t\geq 0$, $\Gamma(t)$ and Γ_0 are both simple closed curves. Moreover, they are smooth (of class C^∞) in x, t.

(A2) There exists $\Omega(r_0, r_1) = \{x\in\mathbf{R}^2; 0<r_0<|x|<r_1\}$ such that $\Omega(r_0, r_1)\subset\Omega(t)$ for all $t\geq 0$. Moreover, there is $\delta > 0$ such that

$$\operatorname{dist}(\Gamma_0, \{|x| = r_0\}) \geq \delta \text{ and}$$
$$\operatorname{dist}(\Gamma(t), \{|x| = r_1\}) \geq \delta \text{ for all } t\geq 0.$$

(A3) $(t+T_p) = \Omega(t)$, $\Gamma(t+T_p) = \Gamma(t)$ and $\beta(\cdot, t+T_p) = \beta(\cdot, t)$ for all $t\geq 0$.

(A4) $g(x)$ is a bounded continuous vector function in $\boldsymbol{R}^2\setminus K$.

(A5) There exists a function $b = b(x, t)$ of the form $b = \operatorname{rot} c(x, t)$ where $c = (x, t)\in C^3$ on $B\times[0,\infty)$, periodic in t with period T_p and $b|_{\partial\Omega(t)} = \beta$.

Remark 1. By (A5), retaking $c(x, t)$, if necessary, it holds

$$\int_{\Gamma_0}\beta\cdot n dS = \int_{\Gamma(t)}\beta\cdot n dS = 0, \text{ where } B \text{ stands for } B_d\setminus K.$$

Here, we state two lemmas.

Lemma 2.1 (cf. Temam [19]). *For an arbitrary $\varepsilon > 0$, there exists $b_\varepsilon = b_\varepsilon(x, t)$ such that*

$$b_\varepsilon\in H^2(B),\ \operatorname{div} b_\varepsilon = 0,\ b_\varepsilon(\partial\Omega(t)) = \beta,$$
$$|((u\cdot\nabla)b_\varepsilon, u)| \leq \varepsilon\|\nabla u\|^2 \text{ for } u\in H^1_\sigma(\Omega(t)).$$

Lemma 2.2 ([12]). *For each $\varepsilon > 0$, there exists $\bar{\theta}_\varepsilon = \bar{\theta}_\varepsilon(x, t)$ such that $\bar{\theta}_\varepsilon\in C(\bar{B})\cap H^2(B)$, $\bar{\theta}_\varepsilon(\Gamma_0) = T_0$, $\bar{\theta}_\varepsilon(\Gamma(t)) = 0$ and $\|(u\cdot\nabla)\bar{\theta}_\varepsilon\| \leq \varepsilon\|\nabla u\|$ for $u\in H^1_0(\Omega(t))$.*

Remark 2. $H^k(B)$ and $H^k_0(B)$ stand for Sobolev spaces. $H_\sigma(B)$ and $H^1_\sigma(B)$ mean solenoidal Sobolev spaces.

Remark 3. Thanks to the assumption (A3), b_ε and $\bar{\theta}_\varepsilon$ can be taken as periodic functions with period T_p.

Proof of Lemma 2.2. To show this lemma, we introduce $\theta_0(x)$:

$$(4)\quad \theta_0(x) = \begin{cases} T_0, & x \in B_{r_0} \setminus K, \\ T_0 \cdot \left(\log \frac{r}{r_1}\right) \Big/ \left(\log \frac{r_0}{r_1}\right), & x \in \Omega(r_0, r_1), \\ 0, & x \in B \setminus B_{r_1}, \end{cases}$$

where $B_{r_i} = \{x \in \boldsymbol{R}^2 ; |x| \leq r_i\}$ $(i = 0,1)$.

On the other hand, according to Lemma 1.9 of Chapter II of Temam [19], for an arbitrary $\varepsilon > 0$, there is $\alpha_\varepsilon = \alpha_\varepsilon(x, t) \in C^2(\Omega(t))$ such that $\alpha_\varepsilon = 1$ in some neighbourhoods of Γ_0 and $\Gamma(t)$; $\alpha_\varepsilon = 0$ if $\rho(x) \geq 2\delta(\varepsilon)$ and $|D_k \alpha_\varepsilon(x)| \leq \varepsilon / \rho(x)$ if $\rho(x) \leq 2\delta(\varepsilon)$ $(k = 1,2)$, where $\rho(x, t) = \min\{\mathrm{dist}(x, \Gamma_0), \mathrm{dist}(x, \Gamma(t))\}$ and $\delta(\varepsilon) = \exp(-1/\varepsilon)$. Now, we put $\bar{\theta}_\varepsilon = \alpha_\varepsilon \theta_0$. Then, thanks to the assumption (A2), we can show, by retaking ε if necessary, the $\bar{\theta}_\varepsilon$ satisfies the condition of Lemma 2.2.

Next, we state an abstract heat convection equation. We start with making the change of variables. We denote $b = b_\varepsilon$ and $\bar{\theta} = \bar{\theta}_\varepsilon$ (Later we retake ε). Then we put

$$u = \hat{u} + b,\ \theta = \hat{\theta} + \bar{\theta},\ (x, y) = d(x^*, y^*),$$
$$t = \frac{d^2 t^*}{\nu},\ \hat{u} = \frac{\nu u^*}{d},\ \hat{\theta} = \frac{\nu T_0 \theta^*}{\kappa},\ p = \frac{\rho \nu^2 p^*}{d^2}.$$

By these relations, we have new variables u^*, θ^*, p^*, x^*, y^*, and t^*. But, after changing variables, we abbreviate asterisks and use the same letters u, θ, p, x, y, and t for the simplicity. Then, equations (1) are transformed to the following:

$$(5)\quad \begin{cases} u_t + (u \cdot \nabla) u = -\nabla p + \Delta u - (u \cdot \nabla) b \\ \qquad - (b \cdot \nabla) u - R\theta - b_t - (b \cdot \nabla) b \\ \qquad + \Delta b + d^3 g/\nu^2 - R(\bar{\theta} - P^{-1}) \text{ in } \hat{\Omega}, \\ \mathrm{div}\, u = 0 \text{ in } \hat{\Omega}, \\ \theta_t + (u \cdot \nabla)\theta = P^{-1}\Delta\theta + P^{-1}\Delta\bar{\theta} \\ \qquad - (u \cdot \nabla)\bar{\theta} - (b \cdot \nabla)\theta - (b \cdot \nabla)\bar{\theta} \text{ in } \hat{\Omega}, \end{cases}$$

$$(6)\quad u|_{\partial\Omega(t)} = 0,\ \theta|_{\Gamma_1} = 0,\ \theta|_{\Gamma(t)} = 0 \text{ for any } t \in (0, \infty),$$

$$(7)\quad u(\cdot, t + T_p) = u(\cdot, t),\ \theta(\cdot, t + T_p) = \theta(\cdot, t) \text{ in } \Omega(t + T_p) = \Omega(t),$$

where $R = \alpha g T_0 d^3 / \kappa\nu$, $P = \nu / \kappa$ and T_p is a period.

Then, we introduce a proper lower semicontinuous convex (p.l.s.c.) function:

$$(8)\quad \varphi_B(U) = \begin{cases} \frac{1}{2} \int_B (|\nabla u|^2 + P^{-1} |\nabla \theta|^2) dx \\ \text{if } U \in H_\sigma^1(B) \times H_0^1(B), \\ + \infty \text{ if } U \in (H_\sigma(B) \times L^2(B)) \setminus \\ (H_\sigma^1(B) \times H_0^1(B)). \end{cases}$$

Here we define a closed convex set $K(t)$ of $H_\sigma(B) \times L^2(B)$ by $K(t) = \{U \in H_\sigma(B) \times L^2(B) ;\ U = 0$ a.e. in $B \setminus \Omega(t)\}$ and denote its indicator function by $I_{K(t)}$, that is, $I_{K(t)}(U) = 0$ if $U \in K(t)$ and $+\infty$ if $U \in (H_\sigma(B) \times L^2(B)) \setminus K(t)$. Then we define another p.l.s.c. function:

(9) $\varphi^t(U) = \varphi_B(U) + I_{K(t)}(U)$ for each $t \in [0, \infty)$ with the effective domain

$$D(\varphi^t) = \{U \in H_\sigma(B) \times L^2(B) ;\ U|_{\Omega(t)} \in H_\sigma^1(\Omega(t)) \times H_0^1(\Omega(t)),\ U|_{B\setminus\Omega(t)} = 0\}.$$

Let $\partial\varphi^t$ be the subdifferential operator of φ^t, then we have:

$$D(\partial\varphi^t) = \{U \in H_\sigma(B) \times L^2(B) ;\ U|_{\Omega(t)} \in (H^2(\Omega(t)) \cap H_\sigma^1(\Omega(t))) \times (H^2(\Omega(t)) \cap H_0^1(\Omega(t))),\ U|_{B\setminus\Omega(t)} = 0\}.$$
$$\partial\varphi^t(U) = \{f \in H_\sigma(B) \times L^2(B) ;\ P(\Omega(t)) f|_{\Omega(t)} = A(\Omega(t)) U|_{\Omega(t)}\}.$$

Here $A(\Omega(t)) = (-P_\sigma(\Omega(t))\Delta, -(1/P)\Delta)$, $P(\Omega(t)) = (P_\sigma(\Omega(t)), 1_{\Omega(t)})$, and $P_\sigma(\Omega(t))$ is a projection $L^2(\Omega(t)) \rightarrow H_\sigma(\Omega(t))$.

Then we have the following abstract heat convection equation AHC in $H_\sigma(B) \times L^2(B)$:

$$(10)\quad \frac{dV}{dt} + \partial\varphi^t(V(t)) + F(t)V(t) + M(t)V(t) \ni P(B) f(t),\ t \in (0, \infty),$$

where $V = (v, \theta)$ and $P(B) = (P_\sigma(B), 1_B)$; moreover

$$F(t)V(t) = (P_\sigma(B)(v \cdot \nabla) v,\ (v \cdot \nabla)\theta),$$
$$M(t)V(t) = (P_\sigma(B)((v \cdot \nabla) b + (b \cdot \nabla) v + R\theta),\ (v \cdot \nabla)\bar{\theta} + (b \cdot \nabla)\theta)$$
$$f = (-b_t - (b \cdot \nabla) b + \Delta b + d^3 g/\nu^2 - R(\bar{\theta} - (1/P)),\ (1/P)\Delta\bar{\theta} - (b \cdot \nabla)\bar{\theta}).$$

Well, we define the strong solution of AHC (see [10]).

Definition 2.3. *Let* $V : [0, S] \mapsto H_\sigma(B) \times L^2(B)$, $S \in (0, \infty)$. *Then* V *is a strong solution of AHC on* $[0, S]$ *if it satisfies the following properties (i) and (ii):*

(i) $V \in C([0, S]; H_\sigma(B) \times L^2(B))$ *and* dV/dt *exists for a.e.* $t \in (0, S]$.

(ii) $V(t) \in D(\partial\varphi^t)$ *for a.e.* $t \in [0, S]$ *and there exists a function* $G : [0, S] \mapsto H_\sigma(B) \times L^2(B)$ *satisfying* $G(t) \in \partial\varphi^t(V(t))$ *and*

$$(11)\quad \frac{dV}{dt} + G(t) + F(t)V(t) + M(t)V(t) = P(B)f(t) \text{ for a.e. } t \in [0, S].$$

Remark 4. If V is a strong solution, then for any $\tau > 0$, both dV/dt and G belong to $L^2(\tau, S; H_\sigma(B) \times L^2(B))$ (see [10]).

Definition 2.4. *A strong solution of AHC is called a periodic strong solution (resp. a strong solution of the initial value problem) if it satisfies the condition* (12) (*resp.* (13)):

$$(12)\quad U(t+T_p) = U(t) \text{ for } t \in [0, \infty) \text{ in } H_\sigma(B) \times L^2(B),$$

$$(13)\quad U(0) = (\tilde{a}, \tilde{h}) \text{ in } H_\sigma(B) \times L^2(B),$$

where $(a, h) \in H_\sigma(\Omega(0)) \times L^2(\Omega(0))$ *and* $\tilde{a}$, $\tilde{h}$ *mean extensions of* a, h *to* B *with putting zero outside* $\Omega(0)$ *respectively.*

3. Results. In our previous paper ([14]), we had a theorem stated as below:

Theorem 3.1. *If physical data, domain constants and the period* T_p *satisfy the relation* (RTP): $(\varepsilon C_1)^{-1}\log(1+2A) \leq T_p$, *then there is a periodic strong solution of AHC. Here* C_1 *is a domain constant,* $\varepsilon > 0$ *is an appropriate small number and* A *does not include* b *but includes* T_p.

Remark 5. $A = 8((4-\varepsilon)C_1)^{-1}(|R|^2 + \|\nabla\bar{\theta}\|^2_{L^\infty(\tilde{B})})(4\kappa/\nu - \varepsilon C_1)^{-1}$.

In this paper, we will improve the above results. Let us make assumptions.

(A 6) $b \in L^\infty(0, \infty; H^2(B))$, $b_t \in L^\infty(0, \infty; L^2(B))$ and $\bar{\theta} \in L^\infty(0, \infty; H^2(B))$.

Theorem 3.2. *If Assumptions* (A1)~(A6) *are satisfied, then the following hold:*

(i) *For sufficiently small* $R = \alpha g T_0 d^3/\kappa\nu$, *there exists a periodic strong solution of AHC with period* T_p.

(ii) *In addition to the above condition, if* $\|b\|_{L^\infty(0,\infty,H^2(B))}$, $\|b_t\|_{L^\infty(0,\infty,L^2(B))}$ *and* $\|\bar{\theta}\|_{L^\infty(0,\infty,H^2(B))}$ *are sufficiently small and* ν *is lorge enough, then the periodic strong solution is unique.*

(iii) *Under the same assumptions on* b, b_t, $\bar{\theta}$ *and* ν, *the periodic strong solution* $U_\pi(t)$ *obtained in* (i) *is asymptotically stable in the following sense, that is,*

$$\|U(t) - U_\pi(t)\|_{L^2(\Omega(t))\times L^2(\Omega(t))} \to 0 \text{ as } t \to \infty,$$

where $U(t)$ *is a strong solution of AHC with* $U(0) = U_\pi(0) + U_0$ *and* U_0 *is an arbitrarily given data in* $H_\sigma(\Omega(0)) \times L^2(\Omega(0))$.

4. Proof of the theorem. We shall state several lemmas.

Lemma 4.1. *There exists a positive constant* C_1 *such that* $\varphi^t(U) \geq C_1\|U\|^2_{L^2(B)}$ *for every* $t \in [0, S]$ *and* $U \in H^1_\sigma(B) \times H^1_0(B)$.

The next lemma is a version of Lemma 2.1 of Foias, Manley and Temam [2].

Lemma 4.2. *Let* $U = (u, \theta)$ *be a strong solution of AHC. Then*

$$(14)\quad \|\theta(t)\|_{L^2(B)} \leq |B|^{1/2}\kappa/\nu + \|\theta(0)\|\exp(-2\kappa t/\nu)$$

holds for $t \in (0, \infty)$, *where* $|B|$ *is a volume of* B.

The following lemma is important.

Lemma 4.3. (i) *Let* $U = (u, \theta)$ *be a strong solution of AHC. Then, for an arbitrary* $\delta \in (0, S)$, *there are positive constants* $a_i(\delta)$ $(i = 1,2,3)$, *independent of S, depending on* b *and* θ, *such that*

$$(15)\quad \varphi^t(U(t)) \leq (a_2(\delta)/\delta + a_3(\delta))\exp(a_1(\delta)) \text{ for any } t \in [\delta, S].$$

(ii) *Furthermore, if* U *is a periodic strong solution with period* T_p, *then the same estimate holds for all* $t \in [0, T_p]$.

Proof of Lemma 4.3. Multiplying AHC by $G(t)$ and integrating on B, then we have for a.e. $t \in (0, S]$

$$(16)\quad \frac{d}{dt}\varphi^t(U(t)) + \|G(t)\|^2 \leq C_4\|U(t)\|^{1/2}\cdot\|U(t)\|_1\cdot\|G(t)\|^{3/2} + |(M(t)U(t), G(t))| + \|f(t)\|\cdot\|G(t)\| + C_2\|G(t)\|\cdot\varphi^t(U(t))^{1/2} + C_3\varphi^t(U(t)),$$

where $\|\cdot\|_k = \|\cdot\|_{H^k(B)}$ and $C_i (i \geq 1)$ are domain constants. From (16), we have

$$(17)\quad \frac{d}{dt}\varphi^t(U(t)) + \frac{1}{2}\|G(t)\|^2 \leq C_5\|U(t)\|^2\varphi^t(U(t))^2 + C_6M_1\varphi^t(U(t)) + (2C_2^2 + C_3)\varphi^t(U(t)) + 2\|f\|^2_{\infty,2},$$

where $M_1 = \|b\|_1\cdot\|b\|_2 + 2\|b\|\cdot\|b\|_2 + \|\bar{\theta}\|_1\cdot\|\bar{\theta}\|_2 + |R|^2$ and $\|f\|_{\infty,2} = \|f\|_{L^\infty(0,\infty;L^2(B))}$. Here we used (3.23) of Chap. Ⅲ in Temam [20]. On the other hand, multiplying AHC by $U(t)$ and integrating on B, then we get

$$(18)\quad \frac{d}{dt}\|U(t)\|^2 + 2C_1\|U(t)\|^2 \leq (4|R|^2/C_1)\|\theta(t)\|^2 + 2\|f\|^2_{\infty,2}/C_1 \text{ for a.e. } t \in (0, S],$$

where we used Lemma 2.1 and Lemma 2.2 with suitable $\varepsilon > 0$. Thanks to Lemma 4.2 and (18),we see for any $t \in (0, S]$

$$(19)\quad \|U(t)\|^2 \leq \exp(-2C_1 t)\|U(0)\|^2 + \{(2|R|^2/C_1^2)(|B|\kappa^2/\nu^2 + \|\theta(0)\|^2) + \|f\|^2_{\infty,2}/C_1^2\}(1 - \exp(-2C_1 t)).$$

Thus, noting $U(t)$ is continuous at $t = 0$, we get an a priori estimate:

(20) $\| U(t) \|^2 \le C_0 + C_0' \| U(0) \|^2$ for any $t \in [0, S]$,

where $C_0 = (2 | R |^2 \cdot | B | \kappa^2/\nu^2 + \| f \|^2_{\infty,2})/C_1^2$ and $C_0' = 1 + 2 | R |^2/C_1^2$.

Making use of (17), (20) and the uniform Gronwall inequality, we get (15), where

$$(21) \quad \begin{cases} a_1(\delta) = (2C_2^2 + C_3 + C_6 M_1)\delta \\ \qquad + C_5(C_0 + C_0' \| U(0) \|^2) a_3(\delta)\delta, \\ a_2(\delta) = 2\delta \| f \|^2_{\infty,2}, \\ a_3(\delta) = 2^{-1}(C_0 + C_0' \| U(0) \|^2) \\ \qquad + (\delta/C_1)\{2 | R |^2 (| B |^2 \kappa^2/\nu^2 \\ \qquad + \| \theta(0) \|^2) + \| f \|^2_{\infty,2}\}. \end{cases}$$

Concerning (ii), we can show by means of the periodicity of $U(t)$ and data. (see [14].)

Lemma 4.4. *If U is a periodic solution, then we have*

$$(22) \quad \| \theta(0) \|^2 \le | B | \kappa^2/\nu^2,$$

$$(23) \quad \| U(0) \|^2 \le (1/C_1^2)(4 | R |^2 \cdot | B | \kappa^2/\nu^2 + \| f \|^2_{\infty,2}).$$

Lemma 4.5. (see [14]). *For any $U_0 = (a, h) \in H(0) \equiv H_\sigma(\Omega(0)) \times L^2(\Omega(0))$, there exists a unique strong solution U of AHC on $[0, S]$ with $U(0) = U_0$.*

Proof of Lemma 4.5. For $U_0 \in H(0)$, there exists a sequence $\{U_{0,n}\} \subset H_\sigma^1(\Omega(0)) \times H_0^1(\Omega(0))$ such that $\| U_{0,n} - U_0 \| \to 0$ as $n \to \infty$. Then we have strong solutions U_n of (AHC) with $U_n(0) = U_{0,n}$ and by means of Gronwall's inequality we get $\| U_n(t) - U_m(t) \|_{H_B} \le C \| U_{0,n} - U_{0,m} \|_{H(0)}$ for all $t \in [0, S]$, where $H_B = H_\sigma(B) \times L^2(B)$ and a constant $C > 0$ is independent of n, m, t. Hence we obtain $U \in C([0, S]; H_B)$ such that $\| U_n(t) - U(t) \|_{H_B} \to 0$ as $n \to \infty$ uniformly on $[0, S]$. Moreover, by viture of (15) and the lower semicontinuity of φ^t, we see $U(t) \in D(\varphi^t)$ for any $t \in (0, S]$. Now, let us fix an arbitrary $\delta \in (0, S)$ and consider an initial value problem as follows:

$$dV/dt + \partial\varphi^t(V(t)) + F(t)V(t) + M(T)V(t) \ni P(B)f(t)$$
$$\text{for } t \in [\delta, S],\ V(\delta) = U(\delta) \in D(\varphi^\delta).$$

Then a unique solution of this problem exists and we get an estimate $\| U_n(t) - V(t) \|_{H_B} \le C \| U_n(\delta) - V(\delta) \|_{H(\delta)}$ for all $t \in [\delta, S]$. Letting $n \to \infty$, then we have $\| U(t) - V(t) \|_{H_B} = 0$ for all $t \in [\delta, S]$. Since $\delta > 0$ is arbitrary and $U \in C([0, S]; H_B)$, therefore we see that U is a strong solution with $U(0) = U_0$.

Proof of Theorem 3.2. To start with, we prove (i) of the theorem. Here we assume $| R | \le C_1/4$. Multiplying AHC by $U(t)$, then we have

$$(24) \quad \frac{1}{2}\frac{d}{dt} \| U(t) \|^2 + 2\varphi^t(U(t))$$
$$\le | ((u \cdot \nabla)b, u) | + | ((b \cdot \nabla)u, u) | + | (R\theta, u) | + | ((u \cdot \nabla)\bar{\theta}, \theta) | + | ((b \cdot \nabla)\theta, \theta) | + | (f(t), U(t)) |$$
$$\le \varepsilon \| \nabla U \|^2 + | R | \cdot \| \theta \| \cdot \| u \| + \varepsilon' \| \nabla\theta \| \cdot \| \nabla u \| + \| f(t) \| \cdot \| U(t) \|$$
$$\le \frac{1}{4}\varphi^t(U(t)) \times 4 + \frac{1}{4\eta} \| f \|^2_{\infty,2},$$

where we used Lemma 2.1 with $\varepsilon = 1/8$ and Lemma 2.2 with $\varepsilon' = (1/8)(\kappa/\nu)^{1/2}$, $\eta = C_1/4 (\ge | R |)$. From (24), we get

$$(25) \quad \frac{d}{dt} \| U(t) \|^2 + 2C_1 \| U(t) \|^2 \le (2/C_1) \| f \|^2_{\infty,2}$$

and we have

$$(26) \quad \| U(t) \|^2 \le \exp(-2C_1 t) \| U(0) \|^2 + (1/C_1^2) \| f \|^2_{\infty,2}(1 - \exp(-2C_1 t)).$$

Here we define a mapping τ as follows:

$$(27) \quad \tau : H = H(0) \equiv H_\sigma(\Omega(0)) \times L^2(\Omega(0)) \to H,$$

$$(28) \quad \tau U(0) = U(T_p) \text{ in } H.$$

Here we used $\Omega(0) = \Omega(T_p)$ and Lemma 4.5. We see τ is continuous in H. Moreover, τ is compact in H, since $\tau U(0) = U(T_p)$ is included in a bounded set of $H_\sigma^1(\Omega(0)) \times H_0^1(\Omega(0))$ by Lemma 4.3. On the other hand, if we taker $r > 0$ such that $(1/C_1) \| f \|_{\infty,2} \le r$, then for $U(0)$ with $\| U(0) \| \le r$ we have by (25)

$$(29) \quad \| U(T_p) \|^2 \le (\exp(-2C_1 T_p)) r^2 + r^2(1 - \exp(-2C_1 T_p)) = r^2.$$

Therefore, we see $\tau B_r \subset B_r$, where $B_r = \{\Phi \in H ; \| \Phi \|_H \le r\}$. Hence, by Schauder's fixed point theorem, there exists $V_0 \in H$ such that $\tau V_0 = V_0$.

Next we prove (ii). Let U_π be the periodic strong solution in (i) and U_1 be any periodic solution. Put $W = U_\pi - U_1$, then we have

$$(30) \quad (1/2)(d \| W(t) \|^2/dt) + 2\varphi^t(W(t)) \le C_7 \varphi^t(W(t)) \varphi^t(U_\pi(t))^{1/2} + C_8 N(t) \varphi^t(W(t))$$

for a.e. $t \in [0, T_p]$. Here $N(t) = \| \nabla b(t) \| + \| \nabla\bar{\theta}(t) \| + | R |$. Noticing (ii) of Lemma 4.3, (21), (22), (23), and using the assumptions of (ii) of this theorem, then we find $2 - C_7 \varphi^t(U_\pi(t))^{1/2} - C_8 N(t) > 0$ for $t \in [0, T_p]$. Thus, we can show the uniqueness of the solution for small data.

Finally, we mention the proof of (iii). Let $U(t)$ be a strong solution of AHC with the initial condition $U(0) = U_\pi(0) + U_0$ where $U_0 \in H_\sigma(\Omega(0)) \times L^2(\Omega(0))$. Put $V = U - U_\pi$. Then we have the similar type inequality to (30) on V. Moreover by virtue of the smallness assumptions on data and the periodicity we can take $\lambda \equiv 2 -$

$C_7\varphi^t(U_\pi(t))^{1/2} - C_8N(t) > 0$ and we get $\| V(t) \|^2 \leq \| V(0) \|^2 \exp(-2\lambda C_1 t)$ for any $t \in (0, \infty)$. Hence, we have shown the (exponential) asymptotic stability of the periodic solution U_π.

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