On the Number of Asymptotic Points of Holomorphic Curves

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1. Introduction. Let $f = [f_1, \dots, f_{n+1}]$ be a transcendental holomorphic curve from $\boldsymbol{C}$ into the n dimensional complex projective space $P^n(\boldsymbol{C})$ with a reduced representation

$$(f_1, \dots, f_{n+1}) : \boldsymbol{C} \to \boldsymbol{C}^{n+1} - \{\mathbf{O}\},$$

where n is a positive integer.

We use the following notation:

$$\|f(z)\| = (|f_1(z)|^2 + \cdots + |f_{n+1}(z)|^2)^{1/2}$$

and for a point $\boldsymbol{a} = (a_1, \dots, a_{n+1})$ in $\boldsymbol{C}^{n+1} - \{\mathbf{O}\}$

$$\|\boldsymbol{a}\| = (|a_1|^2 + \cdots + |a_{n+1}|^2)^{1/2},$$

$$(\boldsymbol{a}, f) = a_1 f_1 + \cdots + a_{n+1} f_{n+1},$$

$$(\boldsymbol{a}, f(z)) = a_1 f_1(z) + \cdots + a_{n+1} f_{n+1}(z),$$

$$d(\boldsymbol{a}, f(z)) = |(\boldsymbol{a}, f(z))| / (\|\boldsymbol{a}\| \|f(z)\|).$$

(On the distance "d", see [7], p. 76, where $\|\ \|$ is used instead of d).

The characteristic function $T(r, f)$ of f is defined as follows (see [7]):

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta - \log \|f(0)\|.$$

We note that

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r} = \infty$$

since f is transcendental.

We put

$$\rho = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r},$$

$$\lambda = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and we say that ρ is the order of f and λ the lower order of f.

Let

$$V = \{\boldsymbol{a} \in \boldsymbol{C}^{n+1} : (\boldsymbol{a}, f) = 0\}.$$

Then, V is a subspace of $\boldsymbol{C}^{n+1}$ and $0 \le \dim V \le n - 1$. It is said that f is linearly nondegenerate when $\dim V = 0$ and linearly degenerate otherwise.

For meromorphic functions in $|z| < \infty$ we shall use the standard notation and symbols of the Nevanlinna theory of meromorphic functions ([2]).

For $\boldsymbol{a} \in \boldsymbol{C}^{n+1} - V$, we put

$$N(r, \boldsymbol{a}, f) = N(r, 1/(\boldsymbol{a}, f))$$

and we denote the standard basis of $\boldsymbol{C}^{n+1}$ by $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_{n+1}$.

Let X be a subset of $\boldsymbol{C}^{n+1}$. Then, we say that X is **in general position** if the elements of X are linearly independent when $\# X \le n$ or if any $n + 1$ elements of X are linearly independent when $\# X \ge n + 1$.

The purpose of this paper is to extend a famous result on the number of asymptotic values of meromorphic functions obtained by Ahlfors in [1] to holomorphic curves. By the way, the result in [1] was extended to algebroid functions by Lü Yinian in [5].

2. Definition and lemma. In this section, we first give a definition of asymptotic point to holomorphic curves. Let f be as in Section 1.

Definition 1 (asymptotic point) (see Definition 3 in [6]). A point $\boldsymbol{a}$ of $\boldsymbol{C}^{n+1} - V$ is an asymptotic point of f if and only if there exists a path Γ : $z = z(t)$ $(0 \le t < 1)$ in $|z| < \infty$ satisfying the following conditions:

(i) $\lim_{t\to 1} z(t) = \infty$;

(ii) $\lim_{t\to 1} d(\boldsymbol{a}, f(z(t))) = 0$.

Remark. This definition is a generalization of "asymptotic values" of meromorphic functions.

In fact, let $g = g_2/g_1$ be a transcendental meromorphic function in $|z| < \infty$, where g_1 and g_2 are entire functions without common zeros. Suppose that g has an asymptotic value c along a path L going from a finite point to ∞ and put $\tilde{g} = [g_1, g_2]$.

(i) When $c \ne \infty$, for $\boldsymbol{a} = (-c, 1) \in \boldsymbol{C}^2$,

$$d(\boldsymbol{a}, \tilde{g}(z)) = \frac{|-cg_1(z) + g_2(z)|}{\|\boldsymbol{a}\|(|g_1(z)|^2 + |g_2(z)|^2)^{1/2}} = \frac{|g(z) - c|}{\|\boldsymbol{a}\|(1 + |g(z)|^2)^{1/2}} \to 0$$

as $z \to \infty$ along L;

(ii) when $c = \infty$, for $\boldsymbol{e}_1 \in \boldsymbol{C}^2$,

$$d(\boldsymbol{e}_1, \tilde{g}(z)) = \frac{|g_1(z)|}{(|g_1(z)|^2 + |g_2(z)|^2)^{1/2}} = \frac{1}{(1 + |g(z)|^2)^{1/2}} \to 0$$

as $z \to \infty$ along L.

Let $\boldsymbol{a}$ be a point of $C^{n+1} - V$ such that for any x $(0 < x \leq 1)$ the set

$$D(x\,;\,\boldsymbol{a}) = \{z : d(\boldsymbol{a}, f(z)) < x\},$$

which is open, contains at least one component not relatively compact. Then, let $\sigma(x\,;\,\boldsymbol{a})$ be a function defined on the interval $(0,1]$ satisfying

(i) for each $x \in (0, 1]$, $\sigma(x\,;\,\boldsymbol{a})$ is a component of $D(x\,;\,\boldsymbol{a})$ which is not relatively compact;

(ii) if $x_1 < x_2$, then $\sigma(x_1\,;\,\boldsymbol{a}) \subset \sigma(x_2\,;\,\boldsymbol{a})$.

Definition 2 (asymptotic spot). We call this $\sigma(x\,;\,\boldsymbol{a})$ an asymptotic spot of f corresponding to $\boldsymbol{a}$ (cf. Chapter 4 in [4]).

Considering the fact that if $\sigma_1(x\,;\,\boldsymbol{a}) \cap \sigma_2(x\,;\,\boldsymbol{a}) \neq \phi$ for an $x \in (0, 1]$, then, $\sigma_1(x\,;\,\boldsymbol{a}) = \sigma_2(x\,;\,\boldsymbol{a})$ since $\sigma_1(x\,;\,\boldsymbol{a})$ and $\sigma_2(x\,;\,\boldsymbol{a})$ are components of $D(x\,;\,\boldsymbol{a})$, we give the following

Definition 3. Let $\sigma_1(x\,;\,\boldsymbol{a})$ and $\sigma_2(x\,;\,\boldsymbol{b})$ be two asymptotic spots of f. Then, we say that they are distinct either if $\boldsymbol{a} \neq \boldsymbol{b}$ or if $\boldsymbol{a} = \boldsymbol{b}$ and there exists an $x \in (0, 1]$ such that

$$\sigma_1(x\,;\,\boldsymbol{a}) \cap \sigma_2(x\,;\,\boldsymbol{a}) = \phi.$$

It is readily seen that $\boldsymbol{a}$ is an asymptotic point of f if and only if there exists an asymptotic spot of f corresponding to $\boldsymbol{a}$.

Remark. There can exist more than one asymptotic spots corresponding to a single point. (see Example 1 given below.)

It is well known that a Picard exceptional value of transcendental meromorphic function in $|z| < \infty$ is an asymptotic value. As a generalization of this fact, we have

Proposition. Suppose that $(\boldsymbol{a}, f(z))$ has 0 as a Picard exceptional value. Then, $\boldsymbol{a}$ is an asymptotic point of f (see [6], Theorem 1).

Unlike the case of meromorphic functions, we have no general results on the number of asymptotic points for holomorphic curves. To obtain a result on it, we classify the asymptotic spots of f as follows.

Definition 4. If an asymptotic spot $\sigma(x\,;\boldsymbol{a})$ of f corresponding to $\boldsymbol{a}$ satisfies the following condition:

(*) There exists a positive number δ (< 1) such that for any x $(0 < x < \delta)$, $\sigma(x\,;\,\boldsymbol{a})$ does not contain any zeros of $(\boldsymbol{a}, f)$,

then we say that $\sigma(x\,;\,\boldsymbol{a})$ is **of first kind**; and **of second kind** otherwise.

Let S_f be a set of asymptotic spots of f and put

$$A(S_f) = \{\boldsymbol{a} : \sigma(x\,;\,\boldsymbol{a}) \in S_f\}.$$

Definition 5. We say that the elements of S_f are **distinct in general position** if they are distinct and if $A(S_f)$ is in general position.

Example 1. Let $h = [1, e^z, e^{2z}, \ldots, e^{nz}]$. Then:

(a) When $\theta = 0$,

$$d(\boldsymbol{e}_1, h(re^{i\theta})) = \frac{1}{(1 + e^{2r\cos\theta} + \cdots + e^{2nr\cos\theta})^{1/2}} \to 0 \ (r \to \infty)$$

and when $\theta = \pi$

$$d(\boldsymbol{e}_1, h(re^{i\theta})) \to 1 \ (r \to \infty).$$

(b) For $j = 2, \ldots, n$, when $\theta = 0$ or π

$$d(\boldsymbol{e}_j, h(re^{i\theta})) = \frac{e^{(j-1)r\cos\theta}}{(1 + e^{2r\cos\theta} + \cdots + e^{2nr\cos\theta})^{1/2}} \to 0 \ (r \to \infty)$$

and when $\theta = \pi/2$ or $3\pi/2$

$$d(\boldsymbol{e}_j, h(re^{i\theta})) \to 1/\sqrt{n+1} \ (r \to \infty).$$

(c) When $\theta = \pi$

$$d(\boldsymbol{e}_{n+1}, h(re^{i\theta})) = \frac{e^{nr\cos\theta}}{(1 + e^{2r\cos\theta} + \cdots + e^{2nr\cos\theta})^{1/2}} \to 0 \ (r \to \infty)$$

and when $\theta = 0$

$$d(\boldsymbol{e}_{n+1}, h(re^{i\theta})) \to 1 \ (r \to \infty).$$

These facts show that $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}$ are asymptotic points of h, all asymptotic spots corresponding to them are of first kind and there are two asymptotic spots corresponding to $\boldsymbol{e}_j$ ($j = 2, \ldots, n$) when $n \geq 2$. In this case $\# S_f = 2n$ and $A(S_f) = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}\}$, which is in general position.

Lemma 1. Let $\sigma_1(x\,;\,\boldsymbol{a}_1), \ldots, \sigma_{n+1}(x\,;\,\boldsymbol{a}_{n+1})$ be $n+1$ asymptotic spots of f distinct in general position. Then, there exists a positive number $\delta \in (0, 1)$ such that for any $x \in (0, \delta)$,

$$\cap_{j=1}^{n+1} \sigma_j(x, \boldsymbol{a}_j) = \phi. \tag{1}$$

Proof. Put $S_f = \{\sigma_j(x\,;\,\boldsymbol{a}_j) : j = 1, \ldots, n+1\}$.

(a) The case when $\# A(S_f) = n+1$. Then, by Definition 5, $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n+1}$ are in general position. For $j = 1, \ldots, n+1$, put

$$\boldsymbol{a}_j = (a_{j1}, a_{j2}, \ldots, a_{jn+1})$$

and

$$g_j = (\boldsymbol{a}_j, f) = a_{j1} f_1 + \cdots + a_{jn+1} f_{n+1},$$

then $\det(a_{ij}) \neq 0$ and $f_1, \ldots, f_{n+1}$ can be represented as linear combinations of $g_1, \ldots, g_{n+1}$:

$$f_j = b_{j1} g_1 + \cdots + b_{jn+1} g_{n+1}$$

and we have for any z

(2) $\|f(z)\| \le \sqrt{n+1}(\max_{1\le j\le n+1}\|\boldsymbol{b}_j\|)\|g(z)\|$,

where $g=[g_1,\dots,g_{n+1}]$ and $\boldsymbol{b}_j=(b_{j1},\dots,b_{jn+1})$.

Now, suppose that (1) is false. Then for any $\delta\in(0,1)$, there is an $x\in(o,\delta)$ such that

$$\Omega(x)=\cap_{j=1}^{n+1}\sigma_j(x,\boldsymbol{a}_j)\ne\phi.$$

Let z_x be a point of $\Omega(x)$, then

$$d(\boldsymbol{a}_j,f(z_x))=\frac{|g_j(z_x)|}{\|\boldsymbol{a}_j\|\|f(z_x)\|}<x\ (j=1,\dots,n+1),$$

so that we have

$$\frac{\|g(z_x)\|}{\|f(z_x)\|}<\sqrt{n+1}(\max_{1\le j\le n+1}\|\boldsymbol{a}_j\|)x,$$

which is contrary to (2) since x can be taken arbitrarily near to zero. This implies that (1) must hold.

(b) The case when $\#A(S_f)<n+1$. Then, $\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n+1}$ are not in general position and by Definition 5, there exist at least two identical vectors in $\{\boldsymbol{a}_1,\dots,\boldsymbol{a}_{n+1}\}$ in this case. For example, suppose without loss of generality that $\boldsymbol{a}_1=\boldsymbol{a}_2$. Then, since $\sigma_1(x;\boldsymbol{a}_1)\ne\sigma_2(x;\boldsymbol{a}_2)$, (1) holds by Definition 3.

Let $\sigma(x;\boldsymbol{a})$ be an asymptotic spot of first kind of f. Then, there is a positive number δ such that for any $x\in(0,\delta)$, $\sigma(x;\boldsymbol{a})$ does not contain any zeros of $(\boldsymbol{a},f)$. For $x\in(0,\delta)$, we put

$$u(z)=\begin{cases}\log\|f(z)\|-\log|(\boldsymbol{a},f(z))|+\log\|\boldsymbol{a}\|+\log x & \text{if } z\in\sigma(x;\boldsymbol{a})\\ 0 & \text{otherwise.}\end{cases}$$

Then, $u(z)$ is a non-negative, non-constant and continuous subharmonic function in $|z|<\infty$. Note that $u(z)>0$ in $\sigma(x;\boldsymbol{a})$. There is an r_0 such that for any $r\ge r_0$

$$(|z|=r)\cap\sigma(x;\boldsymbol{a})\ne\phi.$$

Let

$$E(r)=\{\theta: re^{i\theta}\in\sigma(x;\boldsymbol{a})\}\ (r\ge r_0)$$

and we put

$$B(r,u)=\max_{|z|=r}u(z),$$

$$\ell(r)=m(E(r)),$$

$$\theta(r)=\begin{cases}\infty & \text{if } (|z|=r)\subset\sigma(x;\boldsymbol{a}),\\ \ell(r) & \text{otherwise.}\end{cases}$$

Then, the following lemmas hold.

Lemma 2. For any $r\ge 2r_0$

$$\log B(r,u)>\pi\int_{r_0}^{r/2}\frac{1}{t\theta(t)}dt+O(1).$$

Proof. We apply Theorem 8.3 in [3], p. 548 to our $u(z)$ with $k=1/2$. We note that from (8.1.10) in [3], p. 536,

$$\alpha(r)\ge\pi/\theta(r)$$

in our case and we easily obtain our lemma.

Lemma 3. For any r and R satisfying $r_0\le r<R$,

$$B(r,u)\le\frac{R+r}{R-r}\{T(R,f)-N(R,\boldsymbol{a},f)+O(1)\}.$$

In particular, for $2r\ge\max(2r_0,1)$,

(3) $B(r,u)\le 3T(2r,f)+O(1)$.

Proof. For any z such that $|z|<R$, we have the inequality

$$u(z)\le\frac{1}{2\pi}\int_0^{2\pi}u(Re^{i\theta})\frac{R^2-|z|^2}{|Re^{i\theta}-z|^2}d\theta.$$

Let z be a point satisfying

$$u(z)=B(r,u)\ (|z|=r\ge r_0).$$

Then we obtain by using the definition of $u(z)$ and by the fact that $\|\boldsymbol{a}\|\|f(z)\|/|(\boldsymbol{a},f(z))|\ge 1$

$$B(r,u)\le\frac{R+r}{R-r}\frac{1}{2\pi}\int_0^{2\pi}u(Re^{i\theta})d\theta$$

$$\le\frac{R+r}{R-r}\{T(R,f)-N(R,\boldsymbol{a},f)$$

(4) $$+\log\|\boldsymbol{a}\|+\log x\}.$$

In particular, if we take $R=2r\ge\max(2r_0,1)$ in (4), we obtain (3) since

$$N(2r,\boldsymbol{a},f)\ge 0 \text{ for } 2r\ge 1.$$

3. Theorem. Let f be a transcendental holomorphic curve as in Section 1.

Theorem. Let N be the number of asymptotic spots of f which are of first kind and distinct in general position and suppose that λ is finite. Then, we have

$$N\le\begin{cases}n & \text{if } \lambda\le 1/2n,\\ 2n-1 & \text{if } 1/2n<\lambda<1,\\ 2n\lambda & \text{if } 1\le\lambda<\infty.\end{cases}$$

Proof. (a) We first prove that $N\le 2n\lambda+n$. If $N\le n$, there is nothing to prove, so we suppose without loss of generality that $N\ge n+1$. Suppose now that N is finite and let $\sigma_1(x;\boldsymbol{a}_1),\dots,\sigma_N(x;\boldsymbol{a}_N)$ be N asymptotic spots of f which are of first kind and distinct in general position. Then, by Lemma 1 and Definition 4 we can find two positive numbers $x_0(<1)$ and r_0 such that for every $j=1,\dots,N$

(i) $(\boldsymbol{a}_j,f)$ has no zeros in $\sigma_j(x_0;\boldsymbol{a}_j)$;

(ii) $\sigma_j(x_0;\boldsymbol{a}_j)\cap(|z|=r)\ne\phi\ (r\ge r_0)$;

(iii) The intersections of any $n+1$ of $\sigma_1(x_0;\boldsymbol{a}_1),\dots,\sigma_N(x_0;\boldsymbol{a}_N)$ are empty.

Here, we use $u_j(z)$, $\ell_j(r)$, $\theta_j(r)$ and $B(r,u_j)$ for $\sigma_j(x_0;\boldsymbol{a}_j)$ instead of $u(z)$, $\ell(r)$, $\theta(r)$ and

$B(r, u)$ defined for $\sigma(x\,;\, \boldsymbol{a})$ in Section 2 respectively. Then, by (ii)

$$\ell_j(r) > 0 \ (r \geq r_0\,;\, j = 1, \ldots, N)$$

and by (iii)

$$\sum_{j=1}^{N} \ell_j(r) \leq 2n\pi \ (r \geq r_0). \tag{5}$$

From (5) we have for $r \geq r_0$

$$\sum_{j=1}^{N} \int_{r_0}^{r} \frac{\ell_j(t)}{t} dt \leq 2n\pi \log\frac{r}{r_0}. \tag{6}$$

By the Cauchy-Schwarz inequality

$$\int_{r_0}^{r} \frac{\ell_j(t)}{t} dt \int_{r_0}^{r} \frac{dt}{t\ell_j(t)} \geq \left(\int_{r_0}^{r} \frac{dt}{t}\right)^2 = \left(\log\frac{r}{r_0}\right)^2. \tag{7}$$

From (6) and (7) we obtain the inequality

$$\sum_{j=1}^{N} \frac{\log\frac{r}{r_0}}{\int_{r_0}^{r} \frac{dt}{t\ell_j(t)}} \leq 2n\pi \ (r > r_0). \tag{8}$$

Now, let

$$I_j = \{r : (|z| = r) \subset \sigma_j(x_0, \boldsymbol{a}_j)\}$$

and $\chi_j(r)$ be the characteristic function of I_j. Then we have

$$\pi \int_{r_0}^{r} \frac{dt}{t\theta_j(t)} = \pi \int_{r_0}^{r} \frac{dt}{t\ell_j(t)} - \frac{1}{2} \int_{r_0}^{r} \frac{\chi_j(t)}{t} dt. \tag{9}$$

As

$$\frac{1}{2} \int_{r_0}^{r} \frac{\chi_j(t)}{t} dt \leq \frac{1}{2} \log\frac{r}{r_0},$$

we have from Lemma 2, Lemma 3 and (9)

$$\pi \int_{r_0}^{r} \frac{dt}{t\ell_j(t)} \leq \log T(4r, f) + \frac{1}{2}\log\frac{r}{r_0} + O(1). \tag{10}$$

From (8) and (10) we have for $r > r_0$

$$N \log\frac{r}{r_0} \leq 2n \log T(4r, f) + n \log\frac{r}{r_0} + O(1)$$

from which we easily obtain $N \leq 2n\lambda + n$.

Suppose next that N is infinite. Then we can choose $p = [2n\lambda + n] + 1$ asymptotic spots of f which are of first kind and distinct in general position. Applying the above method to those p asymptotic spots, we obtain the inequality

$$p \leq 2n\lambda + n,$$

which is imposible. This means that N is finite and that the following inequality must hold.

$$N \leq 2n\lambda + n.$$

We note here that the inequality $n + 1 \leq N$ results in $\lambda \geq 1/2n$.

(b) We use the same notation as in the proof of (a). Suppose that $N \geq 2n$. Then by (a), $\lambda \geq 1/2$. From (6) we have

$$\sum_{j=1}^{N} \int_{r_0}^{r} \frac{\ell_j(t)}{t}(1 - \chi_j(t)) dt + \sum_{j=1}^{N} 2\pi \int_{r_0}^{r} \frac{\chi_j(t)}{t} dt \leq 2n\pi \log\frac{r}{r_0}. \tag{11}$$

By the Cauchy-Schwarz inequality, we obtain

$$\int_{r_0}^{r} \frac{\ell_j(t)}{t}(1 - \chi_j(t)) dt \int_{r_0}^{r} \frac{1 - \chi_j(t)}{t\ell_j(t)} dt \geq \left(\int_{r_0}^{r} \frac{1 - \chi_j(t)}{t} dt\right)^2. \tag{12}$$

Case 1. For j such that

$$\int_{r_0}^{r} \frac{1 - \chi_j(t)}{t\ell_j(t)} dt > 0 \ (r > r_0),$$

we have

$$\int_{r_0}^{r} \frac{\ell_j(t)}{t}(1 - \chi_j(t)) dt \geq \left(\int_{r_0}^{r} \frac{1 - \chi_j(t)}{t} dt\right)^2 \Big/ \int_{r_0}^{r} \frac{1 - \chi_j(t)}{t\ell_j(t)} dt$$

and by Lemma 2 and Lemma 3

$$\pi \int_{r_0}^{r} \frac{1 - \chi_j(t)}{t\ell_j(t)} dt = \pi \int_{r_0}^{r} \frac{dt}{t\theta_j(t)} \leq \log T(4r, f) + O(1),$$

so that we have for $r \geq r_0$

$$\int_{r_0}^{r} \frac{\ell_j(t)}{t}(1 - \chi_j(t)) dt \geq \frac{\pi\left(\int_{r_0}^{r} \frac{1 - \chi_j(t)}{t} dt\right)^2}{\log T(4r, f) + O(1)}. \tag{13}$$

Case 2. For j such that

$$\int_{r_0}^{r} \frac{1 - \chi_j(t)}{t\ell_j(t)} dt = 0 \ (r > r_0),$$

we have from (12)

$$\int_{r_0}^{r} \frac{1 - \chi_j(t)}{t} dt = 0$$

and so

$$\frac{\left(\int_{r_0}^{r} \frac{1 - \chi_j(t)}{t} dt\right)^2}{\log T(4r, f) + O(1)} = 0. \tag{14}$$

Using (13) and (14) we have for $r \geq r_0$

$$\sum_{j=1}^{N} \int_{r_0}^{r} \frac{\ell_j(t)}{t}(1 - \chi_j(t)) dt \geq \frac{\pi \sum_{j=1}^{N} \left(\int_{r_0}^{r} \frac{1 - \chi_j(t)}{t} dt\right)^2}{\log T(4r, f) + O(1)}. \tag{15}$$

Since

$$\sum_{j=1}^{N} \left(\int_{r_0}^{r} \frac{1 - \chi_j(t)}{t} dt\right)^2 \geq N\left(\log\frac{r}{r_0}\right)^2$$

$$-2\left(\log\frac{r}{r_0}\right)\sum_{j=1}^{N}\int_{r_0}^{r}\frac{\chi_j(t)}{t}dt,$$

we have from (11) and (15) for $r > r_0$

$$(16)\quad \frac{N - 2\sum_{j=1}^{N}\int_{r_0}^{r}\frac{\chi_j(t)}{t}dt/\log\frac{r}{r_0}}{\{\log T(4r, f) + O(1)\}/\log\frac{r}{r_0}} + 2\sum_{j=1}^{N}\int_{r_0}^{r}\frac{\chi_j(t)}{t}dt/\log\frac{r}{r_0} \le 2n.$$

Let $\{r_\nu\}$ be a sequence tending to ∞ as $\nu \to \infty$ such that

$$\lim_{\nu\to\infty}\frac{\log T(4r_\nu, f)}{\log r_\nu} = \lambda.$$

Putting $r = r_\nu$ in (16) and letting $\nu \to \infty$, we have

$$(17)\quad N \le 2n\lambda + 2A(1-\lambda),$$

where

$$A = \limsup_{\nu\to\infty}\sum_{j=1}^{N}\int_{r_0}^{r_\nu}\frac{\chi_j(t)}{t}dt/\log\frac{r_\nu}{r_0}.$$

Here, we note that the following inequality holds:

$$(18)\quad \sum_{j=1}^{N}\chi_j(t) \le n-1.$$

In fact, suppose to the contrary that

$$\sum_{j=1}^{N}\chi_j(t) \ge n.$$

Then, as $N \ge 2n \ge n+1$, for example, for a $t > r_0$ let

$$\chi_1(t) = \cdots = \chi_n(t) = 1.$$

We then have for any $k \ge n+1$

$$\left(\cap_{j=1}^{n}\sigma_j(x_0, \boldsymbol{a}_j)\right) \cap \sigma_k(x_0, \boldsymbol{a}_k) \ne \phi,$$

which contradicts with (iii) in (a). This implies that (18) must hold. It is easy to obtain

$$(19)\quad 0 \le A \le n-1$$

from (18). The inequality $N \ge 2n$, (17) and (19) imply that $\lambda \ge 1$ and when $1 \le \lambda < \infty$, we have $N \le 2n\lambda$ from (17) since $2A(1-\lambda) \le 0$.

Combining the results obtained in (a) and (b) we have our theorem.

Example 2. Let $f = [1, e^{z^m}, e^{2z^m}, \ldots, e^{nz^m}]$, where m is a positive integer. Then, f is a transcendental holomorphic curve such that $\rho = \lambda = m$. As in the case of Example 1, $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}$ are asymptotic points of f, all asymptotic spots corresponding to them are all of first kind and there are m asymptotic spots corresponding to $\boldsymbol{e}_1$ and to $\boldsymbol{e}_{n+1}$ respectively, $2m$ asymptotic spots corresponding to $\boldsymbol{e}_j$ for each $j = 2, \ldots, n$. In this case, $N = \# S_f = 2nm$ and $A(S_f) = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}\}$, which is in general position.

Remark. (I) When $n = 1$, this theorem corresponds to a famous theorem of Ahlfors in [1] and when $n \ge 2$ this theorem is better than Theorem 2 in [5].

(II) Example 2 shows that this theorem is sharp when λ = an integer ≥ 1.

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