A Note on the Diophantine Equation $(m^3 - 3m)^x + (3m^2 - 1)^y = (m^2 + 1)^z$ *)

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Abstract: Let m be a positive integer. In this note, using some elementary methods, we prove that if $2\|m$, $3m^2 - 1$ is an odd prime, then the equation $(m^3 - 3m)^x + (3m^2 - 1)^y = (m^2 + 1)^z$ has only the positive integer solution $(x, y, z) = (2, 2, 3)$.

1. Introduction. Let $\mathbf{Z}, \mathbf{N}$ be the sets of integers and positive integers respectively. Let a, b and c be positive integers with gcd $(a, b) = 1$. In [6], Terai conjectured that the equation

(1) $$a^x + b^y = c^z,\ x, y, z \in \mathbf{N}$$

has at most one solution (x, y, z) with $x > 1$, $y > 1$, $z > 1$. By the results of Scott [4], (1) has at most one solution if $c = 2$ except in two cases $(a, b, c) = (3, 5, 2)$ or $(3, 13, 2)$ and at most two solutions if c is an odd prime. However, in other cases, this problem is far from solved as yet. In this note we consider the case that a, b and c can be expressed as

(2) $$a = m(m^2 - 3),\ b = 3m^2 - 1,\ c = m^2 + 1,$$

where m is a positive integer with $2 \mid m$. Then (1) has a solution $(x, y, z) = (2, 2, 3)$. In this respect, Terai [5] showed that if b is a prime and there exists a prime l such that $l \mid m^2 - 3$ and $3 \mid e$, where e is the order of 2 modulo l, then (1) has only the solution $(x, y, z) = (2, 2, 3)$. In this note, using some elementary methods, we prove the following result:

Theorem. Let a, b and c be positive integers satisfying (2). If $2 \| m$ and b is an odd prime, then (1) has only the solution $(x, y, z) = (2, 2, 3)$.

2. Preliminaries. Lemma 1([3, pp. 122-124]). Every solution (X, Y, Z, n) of the equation

(3) $$X^2 + Y^2 = Z^n,\ X, Y, Z, n \in \mathbf{Z},\ gcd\,(X, Y) = 1,\ Z > 1,\ n > 1$$

can be expressed as

$$X + Yi = i^r(u + vi)^n,\ Z = u^2 + v^2,\ u,v \in \mathbf{Z},\ gcd\,(u, v) = 1,\ i = \sqrt{-1},\ r \in \{0, 1, 2, 3\}.$$

Lemma 2 ([2] and [5]). The equation

(4) $$1 + X^2 = 2Y^n,\ X, Y, n \in \mathbf{N},\ X > 1,\ n > 2$$

has the only solution $(X, Y, n) = (239, 13, 4)$.

Lemma 3 ([1]). Let $\varepsilon = u + vi$ and $\bar{\varepsilon} = u - vi$, where u, v are nonzero integers with $gcd\,(u, v) = 1$. Further let

(5) $$E(s) = \frac{\varepsilon^s + \bar{\varepsilon}^s}{2u},\ F(s) = \frac{\varepsilon^s - \bar{\varepsilon}^s}{2vi},\ s \in \mathbf{N}.$$

Then $E(s)$, $F(s)$ are integers satisfying $(E(s))^2 + (F(s))^2 = (u^2 + v^2)^s$. Let P be an odd prime, and let s_0 be the least positive integer such that $P \mid F(s_0)$. If $p^{t_0} \| F(s_0)$ and $p^{t_0+t} \mid F(s)$, where s_0, s, t_0, t are positive integers, then we have $s_0 p^t \mid s$.

3. Proof of theorem. If $m = 2$, we see from (2) that $(a, b, c) = (2, 11, 5)$. By [4], then (1) has only the solution $(x, y, z) = (2, 2, 3)$. Therefore, we may assume that $m \geq 6$.

Let (x, y, z) be a solution of (1). Then from (1) and (2) we get $a^x + b^y \equiv (-1)^y \equiv 1 \equiv c^z \pmod{m}$. Since $m \geq 6$, it implies that y must be even.

If $2 \nmid x$, then from (1) we get $(-a/c) = 1$, where $(*/*)$ is Jacobi's symbol. However, since $2 \| m$ and $m^2 + 1 \equiv 5 \pmod{8}$, we find from (2) that

$$\left(\frac{-a}{c}\right) = \left(\frac{a}{c}\right) = \left(\frac{m(m^2-3)}{m^2+1}\right) = \left(\frac{2}{m^2+1}\right)\left(\frac{m/2}{m^2+1}\right)\left(\frac{m^2-3}{m^2+1}\right) = \left(\frac{2}{m^2+1}\right) = -1,$$

a contradiction. So we have $2 \mid x$ and $2 \mid y$.

Since b is an odd prime, if $2 \mid x$, $2 \mid y$ and $2 \mid z$, then we have $c^{z/2} + a^{x/2} = b^y$ and $c^{z/2} - a^{x/2} = 1$. It implies that

(6) $$1 + (b^{y/2})^2 = 2c^{z/2}.$$

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We see from (6) that $(X, Y, n) = (b^{y/2}, c, z/2)$ is a solution of (4). Hence, by Lemma 2, we obtain $z/2 \leq 2$. Then, by (2) and (6), we get

$$3m^4 > 2(m^2+1)^2 = 2c^2 \geq 2c^{z/2} = 1 + b^y > b^2 = (3m^2-1)^2 > 4m^4,$$

a contradiction. So we have $2 \mid x$, $2 \mid y$ and $2 \nmid z$.

If $2 \mid x$, $2 \mid y$ and $2 \nmid z$, then $(X, Y, Z, n) = (a^{x/2}, b^{y/2}, c, z)$ is a solution of (3) with $2 \nmid z$. By Lemma 1, we get

(7) $a^{x/2} + b^{y/2} i = \lambda_1(u + \lambda_2 vi)^2,\ \lambda_1, \lambda_2 \in \{-1, 1\},$

where u, v are positive integers satisfying

(8) $c = u^2 + v^2,\ gcd(u, v) = 1.$

From (7), we get

(9) $b^{y/2} = \lambda_1\lambda_2 v \sum_{j=0}^{(z-1)/2} (-1)^j \binom{z}{2j+1} u^{z-2j-1} v^{2j}.$

Notice that b is an odd prime. We see from (9) that $v = b^k$, where k is an integer with $0 \leq k \leq y/2$. If $k > 0$, then from (2) and (8) we get $m^2 + 1 = c = u^2 + v^2 \geq 1 + b^2 = 1 + (3m^2-1)^2 > 4m^4$, a contradiction. It implies that $k = 0$ and $v = 1$. Therefore, by (2) and (8), we get $u = m$, and by (7),

(10) $a^{x/2} = \lambda_1 m \sum_{j=0}^{(z-1)/2} (-1)^j \binom{z}{2j} m^{z-2j-1},$

(11) $b^{y/2} = \lambda_1\lambda_2 \sum_{j=0}^{(z-1)/2} (-1)^j \binom{z}{2j+1} m^{z-2j-1}.$

Since $2 \parallel m$ and the sum in (10) is odd, we see from (10) that $x = 2$ and

(12) $a = \lambda_1 m \sum_{j=0}^{(z-1)/2} (-1)^j \binom{z}{2j} m^{z-2j-1}.$

Therefore, if $y = 2$, then we obtain $(x, y, z) = (2, 2, 3)$.

We now suppose that $y > 2$. Let $\varepsilon = \lambda_1(m + \lambda_2 i)$ and $\bar{\varepsilon} = \lambda_1(m - \lambda_2 i)$. Then from (2) we get $\varepsilon\bar{\varepsilon} = c$. Further let $E(s)$, $F(s)$ satisfy (5). By Lemma 3, $E(s)$ and $F(s)$ are integers satisfying

(13) $(E(s))^2 + (F(s))^2 = c^s,\ s \in \boldsymbol{N}.$

We see from (2), (11) and (12) that

(14) $E(3) = \lambda_1 a,\ E(z) = a,$

(15) $F(1) = \lambda_1\lambda_2,\ F(2) = 2\lambda_2 m,\ F(3) = \lambda_1\lambda_2 b,\ F(z) = b^{y/2}.$

Since $y/2 > 1$, by Lemma 3, we obtain from (15) that

(16) $z = 3b^{y/2-1} z_1,\ z_1 \in \boldsymbol{N}.$

Therefore, by (13), (14), (15) and (16), we get

(17) $$2b^y > a^2 + b^y = c^z \geq c^{3b^{y/2-1}} > e^{3b^{y/2-1}} > \frac{1}{4!}(3b^{y/2-1})^4 > 3b^{2y-4},$$

where we get $2b^{2y} \geq 2b^{y+4} > 3b^{2y}$, a contradiction. Thus, the theorem is proved.

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