Remarks on Takase's Paper "A Generalization of Rosenhain's Normal Form with an Application"

By Shoji KOIZUMI

Faculty of Engineering, Seikei University

(Communicated by Shokichi IYANAGA, M. J. A., Jan. 13, 1997)

In his paper [2], Takase gave a formula expressing the cross-ratios $\dfrac{a_k - a_l}{a_k - a_m}$ of the branch points (a_k, ∞, a_l, a_m) of a hyperelliptic curve C in terms of the "theta-constants" $\vartheta[\eta](\Omega, 0)$ ([2], Theorem 1.1). This was proved by using Theorem 7.1 (Frobenius' theta relation) and Theorem 8.1 (Thomae's formula) in [1]. Here we shall remark that this formula is immediately derived from Theorem 7.6 in [1], and that this theohem is a direct consequence of Theorem 5.3 in [1], so that Frobenius' relation is not needed in our proof. Then we close the note with two corollaries. In this note we generally follow the assumptions, the definitions and the notations in [2]; but for the convenience, we recall the notations.

A positive integer g is fixed throughout the note, and $B = \{1,2, \ldots, 2g+1\}$. The hyperelliptic curve C is defined by the equation: $s^2 = (t - a_1) \cdots (t - a_{2g+1})$, where a_k, $k = 1,2,\ldots, 2g+1$, are distinct complex numbers. The points P_k of C lying over a_k, $1 \le k \le 2g+1$, and $P_\infty \in C$ over the point ∞ of the Riemann sphere form together the set of branch points of C. The ordered set $(a_1, a_2, \ldots, a_{2g+1}, \infty)$ determines the standard basis of the 1-dimensional homology group $H_1(C, \boldsymbol{Z})$, the corresponding basis $(\omega_1, \omega_2, \ldots, \omega_g)$ of the space of holomorphic differentials on C, and the period matrix Ω of C belonging to the Siegel upperhalf space of genus g. For each $k \in B \cup \{\infty\}$, a numerical vector $\eta_k = \begin{pmatrix} \eta'_k \\ \eta''_k \end{pmatrix} \in \frac{1}{2}\boldsymbol{Z}^{2g}$ is defined by $(\Omega\ 1_g)\eta_k = {}^t\left(\int_{P_\infty}^{P_k} \omega_1, \cdots, \int_{P_\infty}^{P_k} \omega_g\right)$, and the subset $U = \{1,3,\ldots, 2g+1\}$ is characterized by $U = \{k \in B \mid \boldsymbol{e}(2\,{}^t\eta'_k\eta''_k) = 1\}$. We write $\boldsymbol{e}(*) = \exp(2\pi i\, *)$. For two subsets T and S of $B \cup (\infty)$ we write $T \circ S = T \cup S - T \cap S$ and $\eta_T = \sum_{k \in T} \eta_k$; and then we have $\eta_{T\circ S} \equiv \eta_T + \eta_S \bmod \boldsymbol{Z}^{2g}$. We denote the theta constant $\vartheta[\eta_T](\Omega, 0)$ by $\vartheta[T]$. The vector η_k has a sense only upto $\bmod \boldsymbol{Z}^{2g}$, and hence $\vartheta[T]$ is not but $\vartheta[T]^2$ is really meaningful.

The following is the formula in ([2], Thm. 1.1), in spite of a slight difference in appearance.

Theorem 1. *For any* $V_2 \subset B - \{k, l, m\}$, *with* $\# V_2 = g - 1$, *we have*

$$(1)\qquad \frac{a_k - a_l}{a_k - a_m} = \boldsymbol{e}(2\,{}^t\eta'_{\{l,m\}}\eta''_k) \times \frac{\vartheta[U\circ(V_2 \cup \{k, l\})]^2 \vartheta[U\circ(V_2 \cup \{m\})]^2}{\vartheta[U\circ(V_2 \cup \{m, k\})]^2 \vartheta[U\circ(V_2 \cup \{l\})]^2}.$$

Now we need the following formula in ([1], Thm. 7.6, p.3.113).

Lemma 2. *For* $k \in B$ *there is a nonzero-constant* $c_k \in \boldsymbol{C}^\times$, *depending only on the hyperelliptic curve* C *such that for any* $V_1 \subset B - \{k\}$, *with* $\# V_1 = g$, *we have the formula,*

$$(2)\qquad c_k = \boldsymbol{e}(2\,{}^t\eta'_{V_1}\eta''_k) \prod_{i \in V_1} (a_k - a_i) \times \left(\frac{\vartheta[U\circ V_1]}{\vartheta[U\circ(V_1 \cup \{k\})]}\right)^2.$$

This formula (and hence, Theorem 7.6 in [1] also) is an easy combination of the formula (3) in ([1], Thm. 5.3, p.3.81) under the substitution $D = \sum_{i \in V_1} P_i$, and a familiar relation between $\vartheta[\xi + \eta](\Omega, z)$ and $\vartheta[\xi](\Omega, z + (\Omega\ 1_g)\eta)$.

To prove the formula (1) we have only to apply (2) to $\dfrac{\vartheta[U\circ(V_2 \cup \{k, l\})]^2}{\vartheta[U\circ(V_2 \cup \{l\})]^2}$ (and $\dfrac{\vartheta[U\circ(V_2 \cup \{m\})]^2}{\vartheta[U\circ(V_2 \cup \{m, k\})]^2}$, resp.) by substituting $V_2 \cup \{l\}$ by V_1 (and $V_2 \cup \{m\}$ by V_1, resp.).

We take this opportunity to present two corollaries, which are almost direct consequences of theorem 1.

Corollary 3. *Under the same assumptions and notations as in* theorem 1 *we have,*

$$(3.0)\qquad \boldsymbol{e}(2\,{}^t\eta'_{\{l,m\}}\eta''_k)\boldsymbol{e}(2\,{}^t\eta'_{\{m,k\}}\eta''_l)\boldsymbol{e}(2\,{}^t\eta'_{\{k,l\}}\eta''_m) = -1.$$

$$\frac{\boldsymbol{e}(2{}^t\eta'_{\{k,l\}}\eta''_m)\vartheta[U\circ(V_2\cup\{k,l\})]^2\vartheta[U\circ(V_2\cup\{m\})]^2}{a_k-a_l} \tag{3}$$
$$=\frac{\boldsymbol{e}(2{}^t\eta'_{\{l,m\}}\eta''_k)\vartheta[U\circ(V_2\cup\{l,m\})]^2\vartheta[U\circ(V_2\cup\{k\})]^2}{a_l-a_m}$$
$$=\frac{\boldsymbol{e}(2{}^t\eta'_{\{m,k\}}\eta''_l)\vartheta[U\circ(V_2\cup\{m,k\})]^2\vartheta[U\circ(V_2\cup\{l\})]^2}{a_m-a_k}.$$

In fact, the equality (3) is a combination of (1) and (3.0), and (3.0) is straightforward from the equality $\boldsymbol{e}(2({}^t\eta'_k\eta''_l-{}^t\eta'_l\eta''_k))=-1$, and the likes.

Corollary 4. *For $V_2\subset B-\{k_1,k_2,k_3,k_4\}$ with $\# V_2=g-1$, we put $\langle k_i,k_j\rangle=\boldsymbol{e}(2{}^t\eta'_{k_i}\eta''_{k_j})$ $\vartheta[U\circ(V_2\cup\{k_j,k_j\}]^2$. Then we have*

$$\frac{\langle k_1,k_3\rangle}{\langle k_1,k_4\rangle}:\frac{\langle k_2,k_3\rangle}{\langle k_2,k_4\rangle}=\frac{a_{k_1}-a_{k_3}}{a_{k_1}-a_{k_4}}:\frac{a_{k_2}-a_{k_3}}{a_{k_2}-a_{k_4}} \tag{4}$$

Note 4.1. In the formula(1), $\vartheta[U\circ(V_2\cup\{m,\infty\})]^2$ may be written more naturally than $\vartheta[U\circ(V_2\cup\{m\})]^2$, and the formula (1) is a special case of (4).

Note 4.2. Thus we have struck the branch point ∞ out in the formula (4), and this formula is also valid for hyperelliptic curves having no branch point at infinity.

References

[1] D. Mumford: Tata lectures on theta II. Progress in Math., vol. 43, Birkhäuser (1984).

[2] K. Takase: A generalzation of Rosenhain's normal form for hyperelliptic curves with an application. Proc. Japan Acad., **72A**, 162–165 (1996).