Random Media with Many Small Robin Holes

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Let M be a bounded region in $\boldsymbol{R}^2$ with smooth boundary ∂M. Let $B(\varepsilon ; w)$ be the disk of radius ε with the center w. Fix $\sigma \in (0,1)$. Fix α. Let $m = 1,2,\cdots$ be a parameter. We put $n = [m^{1-\sigma}]$. We remove n disks of centers $w(m) = (w_1, \cdots, w_n)$ with radius α/m from M and we get $M_{w(m)} = M \setminus \overline{n \text{ disks}}$. We consider M as a probability space by fixing a positive continuous function V on $\bar{M}$ satisfying

$$\int_M V(x)\,dx = 1$$

so that

$$P(x \in A) = \int_A V(x)\,dx.$$

Let M^n be the product probability space. All configuration M^n of the center of disks $w(m)$ can be considered as a probability space M^n by the statistical law stated above.

We put $\tilde{M}^n = \{w(m) \in M^n ; \overline{B(\alpha/m ; w_i)} \cap \overline{B(\alpha/m ; w_j)} = \phi$ for $i \neq j$, $\overline{B(\alpha/m ; w_i)}$ does not intersect $\partial M\}$. For $\sigma \in (0,1)$, it is easy to show that

$$\lim_{m\to\infty} P(w(m) \in M^n ; w(m) \in \tilde{M}^n) = 1.$$

Hereafter we assume that $w(m) \in \tilde{M}^n$. Let $\mu_j(w(m))$ be the j th eigenvalue of the Laplacian of the following problem:

$$\text{(1.1)} \quad \begin{aligned} -\Delta u(x) &= \lambda u(x) \quad x \in M_{w(m)} \\ u(x) &= 0 \qquad\quad x \in \partial M \\ u(x) + k(\alpha/m)^{\sigma} \frac{\partial}{\partial \nu_x} u(x) &= 0 \\ x \in \cup_{i=1}^{n} \partial B(\alpha/m ; w_i)&. \end{aligned}$$

Here k denotes the positive constant and $\frac{\partial}{\partial \nu_x}$ denotes the derivative along the exterior normal direction with respect to $M_{w(m)}$. Let $\mu_j(V)$ be the j th eigenvalue of the Schrödinger operator $-\Delta + 2\pi k^{-1} \alpha^{1-\sigma} V(x)$ in M under the Dirichlet condition on ∂M. We have the following

Theorem 1. *Fix j. Fix $\sigma \in (0,1)$. Fix an arbitrary $\mu^* > 0$. And we fix an arbitrary $\tilde{\varepsilon} > 0$. Then, there exists a small constant α_0 such that we have*

$$\lim_{m\to\infty} P(w(m) \in M^n ; |\mu_j(w(m)) - \mu_j(V)| < m^{\mu^*}(m^{\sigma-1} + m^{-\sigma})) = 1$$

for $\alpha \in (0, \alpha_0)$.

Remark. It should be remarked that our problem is different from the eigenvalue problem of the Laplacian in a domain with many small Dirichlet balls.
See Kac [2], Rauch-Taylor [5], Ozawa [3],[4]. See also Chavel-Feldman [1], Sznitman [6].

We introduce an operator. Here we write w_i as i. We define

$$r(x, y ; w(m)) = G(x, y) + g_1(\alpha/m) \sum_{i=1}^{s} G(x, i) G(i, y) + \sum_{s=2}^{m^*} g_s(\alpha/m) \textstyle\sum_{(s)} G(x, i_1) G_I G(i_s, y)$$

where $m^* = [(\log m)^2]$. Here the sum $\sum_{(s)}$ is the summation whose indices run over all $i_1, \cdots, i_s$ such that $i_\nu \neq i_\mu$ for $\nu \neq \mu$. Here
$g_s(\varepsilon) = (-1)^s(-(2\pi)^{-1}\log\varepsilon + k(2\pi)^{-1}\varepsilon^{\sigma-1})^{-s}$.

Our proof of Theorem 1 can be obtained by Theorems 2,3 and 4.

We put

$$(\boldsymbol{G}_{w(m)} f)(x) = \int_{M_{w(m)}} G_{w(m)}(x, y) f(y)\,dy$$

and

$$(R_{w(m)} f)(x) = \int_{M_{w(m)}} r(x, y ; w(m)) f(y)\,dy.$$

Then, we have the following

Theorem 2. *There exists $\alpha_0 > 0$ such that (1) holds for any $\alpha \in (0, \alpha_0)$:*

(1) $P(w(m) \in M^n ; \| \boldsymbol{G}_{w(m)} - \boldsymbol{R}_{w(m)} \|_{L^2(M_{w(m)})} \leq C m^{\rho}(m^{-\sigma} + m^{\sigma-1})) \geq 1 - m^{-\xi}$

for some $\xi > 0$. Here ρ is an arbitrary fixed positive number.

We put χ as the characteristic function of $M_{w(m)}$ and

$$(\tilde{\boldsymbol{R}}_{w(m)} f)(x) = \int_M r(x, y ; w(m)) f(y)\,dy.$$

Then, we have the following

Theorem 3. *Fix $\xi > 0$. Then,*

$$P(w(m) \in M^n ; \| \tilde{\boldsymbol{R}}_{w(m)} - \chi \tilde{\boldsymbol{R}}_{w(m)} \chi \|_{L^2(M)} = 0(m^{\xi-\sigma})) \geq 1 - m^{-\xi/2}$$

for $\alpha \in (0, \alpha_0)$. *Here* $\xi > 0$ *is an arbitrary fixed number.*

Let A be the Green operator of $-\Delta + 2\pi k^{-1}\varepsilon^{\sigma-1}V$ in M under the Dirichlet condition on ∂M. Then, we have the following

Theorem 4. *Fix* $\xi > 0$. *Then, there exists a constant* α_0 *independent of* ξ, m *such that*

$$P(w(m) \in M^n; \| \tilde{R}_{w(m)} - A \|_{L^2(M)} \leq m^{\xi+\sigma-1}(\log m)^4) \geq 1 - m^{-\xi/2}.$$

Summing up Theorems 2,3 and 4 we get the desired Theorem 1.

References

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