Fundamental Solution of Anisotropic Elasticity

By Gen NAKAMURA*) and Kazumi TANUMA**)

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Abstract: A formula for the fundamental solution of three-dimensional anisotropic elasticity is given in terms of the eigenvectors and/or the generalized eigenvectors of the associated six-dimensional eigenvalue problem called Stroh's eigenvalue problem. From this formula an explicit closed form of the fundamental solution for transversely isotropic media is obtained.

1. Introduction. The aim of the present paper is to give an explicit formula of the fundamental solution of three-dimensional anisotropic elasticity. Let $C = (C_{ijkl})_{1\le i,j,k,l\le 3}$ be a three-dimensional homogeneous linear anisotropic elastic tensor which satisfies the following symmetry and strong convexity conditions;

(A-1) $C_{ijkl} = C_{klij}$ $(1 \le i, j, k, l \le 3)$

(A-2) $\exists \delta > 0;\ \sum_{i,j,k,l=1}^{3} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \ge \delta \sum_{i,j=1}^{3} \varepsilon_{ij}^2$

for a real matrix $\mathscr{E} = (\varepsilon_{ij})$.

Let $x \in R^3$ and let $G_{km} = G_{km}(x)$ be a solution to

$$\sum_{j,k,l=1}^{3} C_{ijkl}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}G_{km} + \delta_{im}\delta(x) = 0 \text{ in } R^3$$
$$(1 \le i, m \le 3)$$

where (x_1, x_2, x_3), δ_{im} and $\delta(x)$ are the cartesian coordinates of x, the Kronecker delta symbol and the Dirac delta function, respectively. $\boldsymbol{G} = \boldsymbol{G}(x) = (G_{km}\ ;_{m\to 1,2,3}^{k\downarrow 1,2,3})$ is called the fundamental solution to the system of the equations of anisotropic elastostatics. Physically, the solution G_{km} describes the displacement at the point x in the x_k direction due to a point force at the origin in the x_m direction.

Bernett [1] gave a formula for $\boldsymbol{G}$ in terms of an integral on the interval $[0,2\pi]$ whose integrand was a smooth periodic function with a period 2π. Malén [5] gave another formula prior to [1]. That is he shows that $\boldsymbol{G}$ can be expressed in terms of the eigenvectors of Stroh's eigenvalue problem provided that all the eigenvalues are distinct. Malén's formula is useful for the perturbation argument for the fundamental solution (cf. Malén and Lothe [6], Nishioka and Lothe [9,10]) and the estimation of the displacement field and the stress field around a straight dislocation (cf. Malén [4,5]). However, the assumption of distinctness of the eigenvalues is too strict to hold for most crystals, since they have some symmetries.

In this paper we give a formula for $\boldsymbol{G}$ in terms of the eigenvectors and/or the generalized eigenvectors of Stroh's eigenvalue problem, which is slightly different from that of [5], without assuming distinctness of the eigenvalues. As a byproduct we give an explicit closed form of $\boldsymbol{G}$ for transversely isotropic media, because the explicit formulae of the eigenvectors and/or the generalized eigenvectors for the associated eigenvalue problem are available in the case of transversely isotropic media. The explicit closed form of the fundamental solution will be useful for computing the displacement and the stress fields in the elastic medium by the boundary element method, which is an effective method in numerical analysis derived from the integral equation methods for boundary value problems.

2. Result. Let $x \neq 0$. Write

$$\frac{x}{|x|} = (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$$

in terms of the polar coordinates (r, φ, θ) $(r \ge 0,\ 0 \le \varphi \le \pi,\ 0 \le \theta < 2\pi)$. Let

$\boldsymbol{v}^0 = (\sin\theta,\ -\cos\theta,\ 0)$,

$\boldsymbol{w}^0 = (\cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi)$.

Define $\boldsymbol{Q} = \boldsymbol{Q}(\varphi, \theta)$, $\boldsymbol{R} = \boldsymbol{R}(\varphi, \theta)$, $\boldsymbol{T} = \boldsymbol{T}(\varphi, \theta)$ by

$$\boldsymbol{Q} = \langle \boldsymbol{v}^0, \boldsymbol{v}^0\rangle,\ \boldsymbol{R} = \langle \boldsymbol{v}^0, \boldsymbol{w}^0\rangle,\ \boldsymbol{T} = \langle \boldsymbol{w}^0, \boldsymbol{w}^0\rangle,$$

where

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle = (\langle \boldsymbol{v}, \boldsymbol{w}\rangle_{ik}\ ;_{k\to 1,2,3}^{i\downarrow 1,2,3}),$$

*) Department of Mathematics, Science University of Tokyo.

**) Division of Mathematical Sciences, Osaka Kyoiku University.

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{ik} = \sum_{j,l=1}^{3} C_{ijkl} v_j w_l$$

for $\boldsymbol{v} = (v_1, v_2, v_3)$, $\boldsymbol{w} = (w_1, w_2, w_3)$. Moreover define $\boldsymbol{N} = \boldsymbol{N}(\varphi, \theta)$ by

$$\boldsymbol{N} = \begin{bmatrix} -\boldsymbol{T}^{-1}\boldsymbol{R}^T & \boldsymbol{T}^{-1} \\ -\boldsymbol{Q} + \boldsymbol{R}\boldsymbol{T}^{-1}\boldsymbol{R}^T & -\boldsymbol{R}\boldsymbol{T}^{-1} \end{bmatrix}$$

and consider Stroh's eigenvalue problem:

$$\boldsymbol{N}\xi = p\xi.$$

According to (A-2) its eigenvalues $p_\alpha (1 \le \alpha \le 6)$ are not real and we renumerate p_α in the following way:

$$p_{\alpha+3} = \overline{p_\alpha}, \quad \operatorname{Im} p_\alpha > 0 \ (1 \le \alpha \le 3).$$

Let $\xi_\alpha = \begin{bmatrix} \boldsymbol{a}_\alpha \\ \boldsymbol{l}_\alpha \end{bmatrix}$ be the eigenvector or the generalized eigenvector associated with the eigenvalue p_α where $\boldsymbol{a}_\alpha, \boldsymbol{l}_\alpha \in \boldsymbol{C}^3$.

Theorem 2.1 (Main Theorem). *Let* $\boldsymbol{G}(x)$ *be*

$$\boldsymbol{G}(x) = \frac{1}{4\pi |x|} (\operatorname{Im}\{\boldsymbol{L}\boldsymbol{A}^{-1}\})^{-1},$$

where $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$, $\boldsymbol{L} = [\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3]$ *and* $\operatorname{Im}\{\cdot\}$ *denotes the imaginary part of a matrix* $\{\cdot\}$, *then* $\boldsymbol{G}(x)$ *is the fundamental solution.*

3. Proof of Theorem 2.1. To start with, we quote a result of [1], which is obtained by the Fourier transformation.

Theorem 3.1. *Let* $\boldsymbol{G}(x)$ *be*

$$\boldsymbol{G}(x) = \frac{1}{8\pi^2 |x|} \int_0^{2\pi} \langle \eta^0(\phi), \eta^0(\phi) \rangle^{-1} d\phi \quad x \ne 0,$$

where $\eta^0(\phi) = \cos\phi\, \boldsymbol{v}^0 + \sin\phi\, \boldsymbol{w}^0$, *then* $\boldsymbol{G}(x)$ *is the fundamental solution.*

Now following Chadwick and Smith [2], we define the Stroh tensors $\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{S}_3$ as follows. Define $\boldsymbol{v}, \boldsymbol{w}$ by

$$\boldsymbol{v} = \cos\phi\, \boldsymbol{v}^0 + \sin\phi\, \boldsymbol{w}^0,$$
$$\boldsymbol{w} = -\sin\phi\, \boldsymbol{v}^0 + \cos\phi\, \boldsymbol{w}^0.$$

Moreover let

$$\boldsymbol{Q}(\phi) = \langle \boldsymbol{v}, \boldsymbol{v} \rangle, \quad \boldsymbol{R}(\phi) = \langle \boldsymbol{v}, \boldsymbol{w} \rangle,$$
$$\boldsymbol{T}(\phi) = \langle \boldsymbol{w}, \boldsymbol{w} \rangle$$

and define $\boldsymbol{N}_1(\phi), \boldsymbol{N}_2(\phi), \boldsymbol{N}_3(\phi)$ by

$$\boldsymbol{N}_1(\phi) = -\boldsymbol{T}^{-1}(\phi)\boldsymbol{R}^T(\phi), \ \boldsymbol{N}_2(\phi) = \boldsymbol{T}^{-1}(\phi),$$
$$\boldsymbol{N}_3(\phi) = \boldsymbol{R}(\phi)\boldsymbol{T}^{-1}(\phi)\boldsymbol{R}^T(\phi) - \boldsymbol{Q}(\phi).$$

Then we define $\boldsymbol{S}_j (j = 1, 2, 3)$ by

$$\boldsymbol{S}_j = \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{N}_j(\phi) d\phi,$$

and we observe that

$$\frac{1}{2\pi} \int_0^{2\pi} \langle \eta^0(\phi), \eta^0(\phi) \rangle^{-1} d\phi = \boldsymbol{S}_2 \tag{3.1}$$

and $\boldsymbol{S}_2$ is invertible.

Theorem 2.1 follows from the following key lemma.

Lemma 3.1.

$$\boldsymbol{S}\xi_\alpha = i\,\xi_\alpha, \quad (1 \le \alpha \le 3) \tag{3.2}$$

where

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_2 \\ \boldsymbol{S}_3 & \boldsymbol{S}_1^T \end{bmatrix}.$$

Proof of Theorem 2.1. From (3.2) it follows that $\boldsymbol{S}_1\boldsymbol{a}_\alpha + \boldsymbol{S}_2\boldsymbol{l}_\alpha = i\,\boldsymbol{a}_\alpha$. Then $(\boldsymbol{S}_1 - \mathrm{i}\,\boldsymbol{I})\boldsymbol{A} = -\boldsymbol{S}_2\boldsymbol{L}$, where $\boldsymbol{I}$ is the 3×3 unit matrix, and $\boldsymbol{L}\boldsymbol{A}^{-1} = i\,\boldsymbol{S}_2^{-1} - \boldsymbol{S}_2^{-1}\boldsymbol{S}_1$. Combining with (3.1) and Theorem 3.1 we obtain Theorem 2.1.

The rest of this section is devoted to the proof of Lemma 3.1. According to the degeneracy of Stroh's eigenvalue problem, we have to consider the following 6 cases.

Case 1 : $p_\alpha (1 \le \alpha \le 3)$ are distinct,
Case 2 : $p_2 = p_3$, $p_1 \ne p_2$, $\dim \operatorname{Ker}(\boldsymbol{N} - p_2\boldsymbol{I}) = 2$,
Case 3 : $p_1 = p_2 = p_3$, $\dim \operatorname{Ker}(\boldsymbol{N} - p_1\boldsymbol{I}) = 3$,
Case 4 : $p_2 = p_3$, $p_1 \ne p_2$, $\dim \operatorname{Ker}(\boldsymbol{N} - p_2\boldsymbol{I}) = 1$,
Case 5 : $p_1 = p_2 = p_3$, $\dim \operatorname{Ker}(\boldsymbol{N} - p_1\boldsymbol{I}) = 2$,
$\dim \operatorname{Ker}(\boldsymbol{N} - p_1\boldsymbol{I})^2 = 3$,
Case 6 : $p_1 = p_2 = p_3$, $\dim \operatorname{Ker}(\boldsymbol{N} - p_1\boldsymbol{I}) = 1$,
$\dim \operatorname{Ker}(\boldsymbol{N} - p_1\boldsymbol{I})^2 = 2$,
$\dim \operatorname{Ker}(\boldsymbol{N} - p_1\boldsymbol{I})^3 = 3$.

Here for an $n \times n$ matrix $\boldsymbol{M}$, $\operatorname{Ker}\boldsymbol{M} = \{u \in \boldsymbol{C}^n ; \boldsymbol{M}u = 0\}$.

For (3.2) partial results can be seen in [2], Lothe and Barnett [3] and Nakamura [7]. However, we have not seen any complete proof of (3.2) dealing with all the 6 cases. In [8] we give a mathematically rigorous and consistent proof of (3.2) which can be applied to all the 6 cases. In this article we show the outline of this proof.

The following basic lemma is necessary for the proof.

Lemma 3.2 ([2]). $\boldsymbol{N}(\phi) = \begin{bmatrix} \boldsymbol{N}_1(\phi) & \boldsymbol{N}_2(\phi) \\ \boldsymbol{N}_3(\phi) & \boldsymbol{N}_1^T(\phi) \end{bmatrix}$ *satisfies*

$$\boldsymbol{N}'(\phi) = -\boldsymbol{I} - \boldsymbol{N}^2(\phi) \tag{3.3}$$

where $\boldsymbol{N}'(\phi) = d\boldsymbol{N}(\phi)/d\phi$ *and* $\boldsymbol{I}$ *is the* 6×6 *unit matrix.*

Associated with the above cases, we state several lemmas.

Lemma 3.3 ([2], [3], [7]). *Let* $p(\phi)$ *be the solution to the Cauchy problem for the Riccati equation*:

$$p'(\phi) = -1 - p^2(\phi), \quad p(0) = p^0 \tag{3.4}$$

with $Im\, p^0 > 0$. *Define* $K(\phi)$ *and* $m(\phi)$ *by*

$$K(\phi) = 2\int_0^{\phi} p(\phi')\, d\phi'$$

and

$$m(\phi) = \int_0^{\phi} \exp[-K(\phi')]\, d\phi'$$

respectively. Then we have

(3.5) $$\int_0^{2\pi} p(\phi)\, d\phi = 2\pi i,$$

(3.6) $$\int_0^{2\pi} \exp[-K(\phi)]\, d\phi = 0$$

and

(3.7) $$\int_0^{2\pi} \exp[-K(\phi)] m(\phi)\, d\phi = 0.$$

Lemma 3.4. *Let ξ be an eigenvector of $\boldsymbol{N}(0)$ associated with an eigenvalue p^0 satisfying $Im\, p^0 > 0$. Let $p(\phi)$ be the unique solution to the Cauchy problem* (3.4). *Then*

$$[\boldsymbol{N}(\phi) - p(\phi)\boldsymbol{I}]\xi = 0.$$

Proof. Put $\boldsymbol{h}(\phi) = [\boldsymbol{N}(\phi) - p(\phi)\boldsymbol{I}]\xi$. From (3.3) and (3.4), $\boldsymbol{h}'(\phi) = -[\boldsymbol{N}(\phi) + p(\phi)\boldsymbol{I}]\boldsymbol{h}(\phi)$. Hence $\boldsymbol{h}(0) = [\boldsymbol{N}(0) - p^0\boldsymbol{I}]\xi = 0$ and the uniqueness of the solution to the Cauchy problem implies $\boldsymbol{h}(\phi) = 0$.

From (3.5) and Lemma 3.4, (3.2) holds for each eigenvector associated with one of the eigenvalues of Case 1 ~ Case 6. Note that in Cases 1,2,3, there are no generalized eigenvectors.

As in the proof of Lemma 3.4, by the uniqueness of the solution to the Cauchy problem we have the next lemma.

Lemma 3.5 ([2], [3]). *Let p^0 be an eigenvalue of $\boldsymbol{N}(0)$ satisfying $Im\, p^0 > 0$. Let ξ_1, ξ_2 be vectors of the Jordan chain of height* 2 *associated with p^0. That is,*

$$[\boldsymbol{N}(0) - p^0\boldsymbol{I}]\xi_1 = 0, \quad [\boldsymbol{N}(0) - p^0\boldsymbol{I}]\xi_2 = \xi_1.$$

Define $\xi_2(\phi)$ as the unique solution to the Cauchy problem:

(3.8) $\xi_2'(\phi) = 2p(\phi)\xi_2(\phi), \quad \xi_2(0) = \xi_2$

where $p(\phi)$ is the unique solution to the Cauchy problem (3.4). *Then*

$$[\boldsymbol{N}(\phi) - p(\phi)\boldsymbol{I}]\xi_2(\phi) = \xi_1.$$

Solving (3.8) and combining with (3.5), (3.6) and Lemma 3.5 we observe that (3.2) holds for each generalized eigenvector ξ_2 when $\{\xi_1, \xi_2\}$ is the Jordan chain of height 2 for Case 4 and Case 5. Note that a Jordan chain of height 2 exists only for Case 4 and Case 5. Also as in the proof of Lemma 3.4, by the uniqueness of the solution to the Cauchy problem we have

Lemma 3.6 ([7]). *Let p^0 be an eigenvalue of $\boldsymbol{N}(0)$ satisfying $Im\, p^0 > 0$. Let ξ_1, ξ_2, ξ_3 be vectors of the Jordan chain of height* 3 *associated with p^0. That is,*

$$[\boldsymbol{N}(0) - p^0\boldsymbol{I}]\xi_1 = 0, \quad [\boldsymbol{N}(0) - p^0\boldsymbol{I}]\xi_2 = \xi_1,$$
$$[\boldsymbol{N}(0) - p^0\boldsymbol{I}]\xi_3 = \xi_2.$$

Moreover let $p(\phi)$ be the unique solution to the Caychy problem (3.4). *Define $\xi_2(\phi)$ and $\xi_3(\phi)$ as the unique solutions to the Cauchy problem*:

(3.9) $\xi_2'(\phi) = 2p(\phi)\xi_2(\phi) - \xi_1, \quad \xi_2(0) = \xi_2$

and

(3.10) $\xi_3'(\phi) = 4p(\phi)\xi_3(\phi), \quad \xi_3(0) = \xi_3$

respectively. Then we have

$$[\boldsymbol{N}(\phi) - p(\phi)\boldsymbol{I}]\xi_2(\phi) = \xi_1$$

and

$$[\boldsymbol{N}(\phi) - p(\phi)\boldsymbol{I}]\xi_3(\phi) = \xi_2(\xi).$$

Solving (3.9), (3.10), and combining with (3.5), (3.6), (3.7) and Lemma 3.6 we observe that (3.2) holds for each generalized eigenvectors ξ_2, ξ_3 if $\{\xi_1, \xi_2, \xi_3\}$ is the Jordan chain of height 3 for Case 6.

Therefore we have proved Lemma 3.1 for any possible eigenvectors and generalized eigenvectors in the Jordan chain arising in Case 1 ~ Case 6.

As a final remark, we point out that the above arguments show that the structure of the Jordan chains remains invariant while ϕ changes.

4. Fundamental solution for transversely isotropic media. Since the surface impedance tensor $-i\boldsymbol{LA}^{-1}$ can be computed explicitly for three-dimensional transversely isotropic media, we give the explicit closed formula for the fundamental solution by using Theorem 2.1. Assume (A-1), (A-2) and $C_{ijkl} = C_{jikl}$. Let the x_3-axis be the axis of rotational symmetry. Then non-zero components of the elastic tensor are characterized by the five independent constants:

$$A = C_{1111} = C_{2222}, \quad C = C_{3333},$$
$$F = C_{1133} = C_{2233}, \quad L = C_{1313} = C_{2323},$$
$$N = C_{1122}, \quad C_{1212} = (A - N)/2.$$

Let $x \neq 0$ and $x/|x| = (\sin\varphi\cos\theta, \sin\varphi \sin\theta, \cos\varphi)$, $(0 \leq \varphi \leq \pi, 0 \leq \theta < 2\pi)$. Define K, G, H, D, Δ, D' by

$$K = \left[\cos^2\varphi + \frac{2L}{A-N}\sin^2\varphi\right]^{1/2},$$

$$G = \left[\frac{2AL\cos^2\varphi + (AC - F^2 - 2FL)\sin^2\varphi + 2\sqrt{AL}\sqrt{\Delta}}{AL}\right]^{1/2},$$

$$H = \left[\frac{\Delta}{AL}\right]^{1/2},$$

$$D = (A\cos^2\varphi + L\sin^2\varphi + AH)K - AG\cos^2\varphi,$$

$$\Delta = AL\cos^4\varphi + (AC - F^2 - 2FL)\cos^2\varphi\sin^2\varphi$$

$$+ CL\sin^4\varphi,$$

$$D' = \{AGHKL - (AC - F^2 - 2FL)\cos^2\varphi \sin^2\varphi\}(\cos^2\varphi + H) - AL\cos^2\varphi(\cos^2\varphi + H)^2 + L\sin^2\varphi(GHKL - C\cos^2\varphi \sin^2\varphi).$$

Then we have

$$G(x) = \frac{1}{4\pi|x|} S_2(\varphi, \theta),$$

$$S_2(\varphi, \theta) = S_2^T(\varphi, \theta) = (S_{ij}(\varphi, \theta)\,;^{i\downarrow 1,2,3}_{j\rightarrow 1,2,3})$$

where

$$S_{11} = \left[\frac{\sin^2\theta}{GK} + \frac{\cos^2\theta}{D'}\left\{AL(GHK - \cos^4\varphi - H\cos^2\varphi) - (AC - (F+L)^2)\cos^2\varphi\sin^2\varphi\right\}\right]\frac{D}{AL\sin^2\varphi},$$

$$S_{12} = \left[\frac{-1}{GK} + \frac{1}{D'}\left\{AL(GHK - \cos^4\varphi - H\cos^2\varphi) - (AC - (F+L)^2)\cos^2\varphi\sin^2\varphi\right\}\right]\frac{D\cos\theta\sin\theta}{AL\sin^2\varphi},$$

$$S_{13} = \frac{(F+L)D}{AD'}\cos\varphi\sin\varphi\cos\theta,$$

$$S_{22} = \left[\frac{\cos^2\theta}{GK} + \frac{\sin^2\theta}{D'}\left\{AL(GHK - \cos^4\varphi - H\cos^2\varphi) - (AC - (F+L)^2)\cos^2\varphi\sin^2\varphi\right\}\right]\frac{D}{AL\sin^2\varphi},$$

$$S_{23} = \frac{(F+L)D}{AD'}\cos\varphi\sin\varphi\sin\theta,$$

$$S_{33} = \frac{D}{AD'}(A\cos^2\varphi + L\sin^2\varphi + AH).$$

In the case of $\sin\varphi = 0$, $S_{ij}(1 \leq i, j \leq 3)$ are obtained by taking the limit $\sin\varphi \rightarrow 0$ in the above formula. Details of the computations of the surface impedance tensor for transversely isotropic media are seen in Tanuma [11].

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