The Structure of Subgroup of Mapping Class Groups Generated by Two Dehn Twists

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(Communicated by Heisuke HIRONAKA, M. J. A., Dec. 12, 1996)

1. Introduction. The mapping class group $\mathcal{M}_{g,n}$ is defined by the set of all orientation-preserving homeomorhpisms of an oriented closed surface $\Sigma_{g,n}$ with genus g and n-punctured. It is an interesting object in topology, and its presentation as a combinatorial group has been determined by Hatcher and Thurston. But the structure of subgroups of mapping class groups has not been sufficiently researched yet.

It is well known that a Dehn twist along a simple closed curve a on $\Sigma_{g,n}$ is defined as an element of $\mathcal{M}_{g,n}$ (see [1]), and we denote it by τ_a. In this paper, it will be shown that the subgroups of mapping class groups generated by two Dehn twists τ_a, τ_b are free groups in general cases.

The minimal intersection number is generally defined for any pair of simple closed curves (a, b) by following, and we denote it by $I_{min}(a, b)$ ([2]).

Definition 1.1. The minimal intersection number $I_{min}(a, b)$ is minimum of the number of $\alpha \cap \beta$ for all α in the isotopy class of a and all β in the isotopy class of b.

Theorem 1.2. $I_{min}(a, b) \geq 2$, then there are no relations between τ_a and τ_b.

Remark 1.3. It is immediately shown that if $I_{min}(a, b) = 0$, then τ_a and τ_b generate an abelian subgroup (i.e. $\tau_a\tau_b = \tau_b\tau_a$). Moreover, it is easily shown that if $I_{min}(a, b) = 1$, then there are two cases:

$$\begin{cases} \langle \tau_a, \tau_b \mid \tau_a\tau_b\tau_a = \tau_b\tau_a\tau_b, (\tau_a\tau_b\tau_a)^4 = 1 \rangle & \text{if (g, n) = (1,0) or (1,1),} \\ \langle \tau_a, \tau_b \mid \tau_a\tau_b\tau_a = \tau_b\tau_a\tau_b \rangle & \text{if otherwise.} \end{cases}$$

In the former case the subgroup is isomorphic to SL(2, Z), and in the latter case the subgroup is isomorphic to 3-strings braid group.

2. Dehn twists and the minimal intersection number. Lemma 2.1. When α, β, and γ are arbitrary three simple closed curves and put $\Gamma = \tau_\alpha^n(\gamma)$ for arbitrary integer n, then

$$|n| * I_{min}(\gamma, \alpha) * I_{min}(\alpha, \beta) - I_{min}(\Gamma, \beta) \leq I_{min}(\gamma, \beta).$$

We denote a tubular neighbourhood of α by N_α, and we can draw Γ so as to coincide with γ on the outside of N_α. Then, we draw Γ' which is isotopic to Γ and is transversely intersecting to γ only one time in each interval in the outside of N_α. The pair of γ and Γ' is a configuration to attain minimum of intersection number (then there exists at least one hyperbolic metric realizing γ and Γ' as geodesics).

We can choose a representative β' in the isotopy class of β such that (A) the pair of β' and γ is a configuration to attain minimum of intersection number and such that (B) the pair of β' and Γ' is a configuration to attain minimum of intersection number. (For example, β' is a geodesic for the metric above.) We can assume the condition (C) that β' satisfies $\beta' \cap \gamma \cap \Gamma' = \emptyset$, because we can isotopically perturb β' to general position with keeping (A) and (B).

One hand, $\gamma \cup \Gamma'$ is an image of some continuous map from $|n| * I_{min}(\gamma, \alpha)$ copies of S^1, and the image from each S^1 is homotopic to α. Then,

$$\#(\beta' \cap (\gamma \cup \Gamma')) \geq |n| * I_{min}(\gamma, \alpha) * I_{min}(\alpha, \beta).$$

On the other hand, from the condition (C),

$$\#(\beta' \cap (\gamma \cup \Gamma')) = \#(\beta' \cap \gamma) + \#(\beta' \cap \Gamma').$$

From (A) and (B),

$$\#(\beta' \cap \gamma) = I_{min}(\gamma, \beta),$$
$$\#(\beta' \cap \Gamma') = I_{min}(\Gamma, \beta).$$

Finally, we get

$$I_{min}(\gamma, \beta) + I_{min}(\Gamma, \beta) \geq |n| * I_{min}(\gamma, \alpha) * I_{min}(\alpha, \beta). \qquad \square$$

Remark 2.2. This proof is partialy almost same as *EXPOSÉ* 4 *Appendice* in [3]. However, Dehn twist is done along only one loop in our observation. Therefore we do not have to assume $n > 0$.

The following lemma was suggested by T. Ohtsuki.

Lemma 2.3. When three simple closed

Proof of Lemma 2.1.

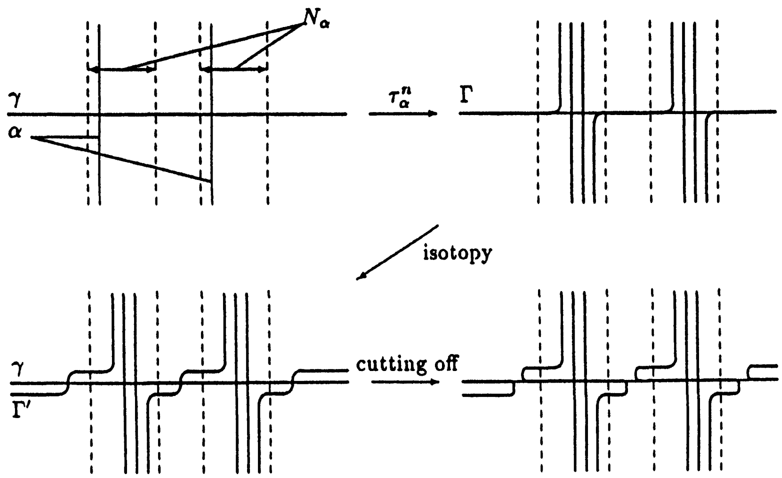

Fig. 1

curves a, b, c satisfy $I_{min}(a, b) \geq 2$, then

$$I_{min}(a, c) > I_{min}(b, c) \Rightarrow I_{min}(a, \tau_a^n(c)) < I_{min}(b, \tau_a^n(c)) \quad \textit{for } \forall n \neq 0.$$

Proof of Lemma 2.3. We can change 'n' to '$-n$' in the lemma, and following two equalities

$$I_{min}(a, \tau_a^{-n}(c)) = I_{min}(\tau_a^n(a), c) = I_{min}(a, c)$$
$$I_{min}(b, \tau_a^{-n}(c)) = I_{min}(\tau_a^n(b), c)$$

can be easily shown by properties of the minimal intersection number. Then we will show an equivalent statement

$$I_{min}(a, c) > I_{min}(b, c) \Rightarrow I_{min}(a, c) < I_{min}(\tau_a^n(b), c) \quad \textit{for } \forall n \neq 0.$$

The following inequality is known by lemma 2.1.

$$|n| * I_{min}(a, b) * I_{min}(a, c) - I_{min}(\tau_a^n(b), c) \leq I_{min}(b, c).$$

Therefore

$$|n| * I_{min}(a, b) * I_{min}(a, c) - I_{min}(b, c) \leq I_{min}(\tau_a^n(b), c).$$

Using the conditions $|n| * I_{min}(a, b) \geq 2$ and $I_{min}(a, c) > I_{min}(b, c)$, we have

$$I_{min}(a, c) < I_{min}(\tau_a^n(b), c). \qquad \square$$

3. Proof of Theorem 1.2. Suppose

$$\tau_b^{m_k}\tau_a^{n_k} \cdots \tau_b^{m_1}\tau_a^{n_1} = 1 \quad (n_i, m_i \neq 0 \quad \textit{for all } i)$$

and it will lead a contradiction.

From the trivial inequality $I_{min}(a, a) < I_{min}(b, a)$, we have

$$I_{min}(a, \tau_a^{n_1}(a)) < I_{min}(b, \tau_a^{n_1}(a)).$$

Using the lemma 2.3 in the following each step,

$$I_{min}(a, \tau_b^{m_1}\tau_a^{n_1}(a)) > I_{min}(b, \tau_b^{m_1}\tau_a^{n_1}(a))$$
$$I_{min}(a, \tau_a^{n_2}\tau_b^{m_1}\tau_a^{n_1}(a)) < I_{min}(b, \tau_a^{n_2}\tau_b^{m_1}\tau_a^{n_1}(a))$$
$$\vdots \qquad\qquad \vdots$$
$$I_{min}(a, \tau_b^{m_k}\tau_a^{n_k}\cdots\tau_b^{m_1}\tau_a^{n_1}(a)) > I_{min}(b, \tau_b^{m_k}\tau_a^{n_k}\cdots\tau_b^{m_1}\tau_a^{n_1}(a))$$

then we have $I_{min}(a, a) > I_{min}(b, a)$, and it is a contradiction. $\square$

References

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[3] A. Fathi, F. Laudenbach, and V. Poenaru: Travaux de Thurston sur les surfaces. Asterisque, **66–67**, pp.68–69 (1979).