A New Version of the Factorization of a Differential Equation of the Form $F(x,y,\tau y)=0$

By Raymond GERARD*) and Hidetoshi TAHARA**)

(Communicated by Shokichi IYANAGA, M. J. A., Nov. 12, 1996)

In this note, we will consider equations of the form

(E$_0$) $F(x, y, \tau y) = 0,$

where $F(x, y, X)$ is a holomorphic function defined in a neighborhood of the origin of $(\boldsymbol{C}_x)^n \times \boldsymbol{C}_y \times \boldsymbol{C}_X$, and τ is a vector field

$$\tau = \sum_{1 \leqslant i \leqslant n} \alpha_i(x, y)\partial/\partial x_i$$

with coefficients $\alpha_i(x, y)(1 \leqslant i \leqslant n)$ meromorphic in x at most with only poles along a union of a finite number of hyperplanes (in $(\boldsymbol{C}_x)^n$) and holomorphic in y near the origin of $(\boldsymbol{C}_x)^n \times \boldsymbol{C}_y$.

If $F(x, y, X)$ is of finite order, say m, with respect to the variable X by Weierstrass preparation theorem $F(x, y, X) = 0$ is equivalent to

$$X^m + \sum_{1 \leqslant j \leqslant m} a_j(x, y)X^{m-j} = 0$$

and (E$_0$) is reduced to

(E) $(\tau y)^m + \sum_{1 \leqslant j \leqslant m} a_j(x, y)(\tau y)^{m-j} = 0.$

In our previous paper [1] we have presented a factorization theorem for (E) which asserts that (E) is factorized into a product of equations of the form $\tau y = f(x, y)$ near the point $x = 0$. In this note we will present a new version of this theorem.

§1. Factorization theorems. Let us consider the following differential equation:

(E) $F(x, y, \tau y) = (\tau y)^m + \sum_{1 \leqslant j \leqslant m} a_j(x, y)(\tau y)^{m-j} = 0$

where $m \in \boldsymbol{N}^*(=\{1, 2, \ldots\})$, $x = (x_1, \ldots, x_n) \in \boldsymbol{C}^n$, $n \in \boldsymbol{N}^*$, $y \in \boldsymbol{C}$, and $a_j(x, y)(1 \leqslant j \leqslant m)$ are holomorphic functions defined in a neighborhood of the origin (0,0) of $(\boldsymbol{C}_x)^n \times \boldsymbol{C}_y$. In (E), $y = y(x)$ is regarded as an unknown function of x and τ is a vector field of the form

$$\tau = \sum_{1 \leqslant i \leqslant n} \alpha_i(x, y)\partial/\partial x_i$$

whose coefficients $\alpha_i(x, y)(1 \leqslant i \leqslant n)$ are meromorphic in x at most with only poles along a union of a finite number of hyperplanes (in $(\boldsymbol{C}_x)^n$) and holomorphic in y in a neighborhood of the origin $(x, y) = (0,0)$ in $(\boldsymbol{C}_x)^n \times \boldsymbol{C}_y$.

Definition 1. We say that the transformation

$$x = (x_1, \ldots, x_n) \rightarrow t = (t_1, \ldots, t_n)$$

is of type (GT) if it is defined by the following: first we transform $x = (x_1, \ldots, x_n) \rightarrow \xi = (\xi_1, \ldots, \xi_n)$ by $x = A\xi$ for some $A \in GL(n, \boldsymbol{C})$ and then we transform $\xi \rightarrow t$ by

$$\xi_1 = (t_1)^k,\ \xi_2 = (t_1)^k t_2, \ldots, \xi_n = (t_1)^k t_n$$

for some $k \in \boldsymbol{N}^*$.

The result of our previous paper [1] is as follows:

Theorem 1 ([Theorem 2.2; 1]). *After a suitable transformation $x \rightarrow t$ which is obtained by a composition of a finite number of transformations of type (GT), we can choose $c \in \boldsymbol{C}$ such that the following conditions hold:*

1) *$c = 0$ or $|c|$ is sufficiently small;*

2) *by setting $y = c + z$ the equation* (E) *is decomposed in a neighborhood of the origin* (0,0) $\in (\boldsymbol{C}_t)^n \times \boldsymbol{C}_z$ *into the form*

(1.1) $\Pi_{1 \leqslant j \leqslant m}(\tau^* z - \varphi_j(t, z)) = 0,$

where τ^* is the transform of τ by the transformation $x \rightarrow t$ and $\varphi_j(t, z)(1 \leqslant j \leqslant m)$ are holomorphic functions defined in a neighborhood of (0,0) $\in (\boldsymbol{C}_t)^n \times \boldsymbol{C}_z$.

Note that the original equation (E) is considered near $(x, y) = (0,0)$; but the decomposition (1.1) is obtained in a neighborhood of $(x, y) = (0, c)$ which may exclude the point $(x, y) = (0,0)$ in case $c \neq 0$. Therefore, if we want to study the behaviour of the solutions of (E) near the origin (0,0) we must fill some gaps between (E) and (1.1).

To fill up the gap we will present here a new version of factorization theorem. In our new result, instead of using transformations of type (GT) and a shift $y = c + z$ we will use the following transformation:

Definition 2. We say that the transformation

$$(x, y) = (x_1, \ldots, x_n, y) \rightarrow (t, z) = (t_1, \ldots, t_n, z)$$

is of type (NGT) if it is defined by the follow-

*) Institut de Recherche Mathématique Alsacien, Université Louis Pasteur, France.

**) Department of Mathematics, Sophia University.

ing: first we transform $x \to t$ by a transformation of type (GT), and then we transform $(t, y) \to (t, z)$ by $y = (t_1)^p z^q$ for some $p, q \in \boldsymbol{N}^*$.

We have the following:

Theorem 2. *After a suitable transformation $(x, y) \to (t, z)$ which is obtained by a composition of a finite number of transformations of type (NGT), the equation* (E) *is decomposed in a neighborhood of the origin $(0,0) \in (\boldsymbol{C}_t)^n \times \boldsymbol{C}_z$ into the form*

$$(\mathrm{E}^*) \qquad \Pi_{1 \leqslant j \leqslant m}(\tau^* z - \varphi_j(t, z)) = 0,$$

where $\tau^* z$ is the transform of τy by the transformation $(x, y) \to (t, z)$ and $\varphi_j(t, z)(1 \leqslant j \leqslant m)$ are holomorphic functions defined in a neighborhood of $(0,0) \in (\boldsymbol{C}_t)^n \times \boldsymbol{C}_z$.

Remarks 1. $\tau^* z$ has the form

$$\tau^* z = \textstyle\sum_{1 \leqslant i \leqslant n} \beta_i(t, z) \partial z / \partial t_i + \beta_0(t, z) z,$$

where $\beta_i(t, z)(0 \leqslant i \leqslant n)$ are meromorphic in t with only poles along a union of a finite number of hyperplanes (in $(\boldsymbol{C}_t)^n$) and holomorphic in z in a neighborhood of $(0,0) \in (\boldsymbol{C}_t)^n \times \boldsymbol{C}_z$.

2. If τ has the form $\tau = \sum_{1 \leqslant i \leqslant n} \alpha_i(x) \partial / \partial x_i$, then

$$\tau^* z = z^{q-1} \times (\textstyle\sum_{1 \leqslant i \leqslant n} \beta_i(t) \partial z / \partial t_i + \beta_0(t) z)$$

for some $q \in \boldsymbol{N}^*$.

3. If $n = 1$ and $\tau = x(d/dx)$, then $\tau^* z = t^p z^{q-1}(at(dz/dt) + bz)$ for some positive numbers a and b.

§2. Sketch of proof. In the proof of Theorem 1 in [1], we used an induction argument. If we notice the lemma given below, we can prove Theorem 2 by induction on m in the same way.

Let $m \geqslant 2$ be an integer, let $a_j(x, y)$ $(2 \leqslant j \leqslant m)$ be holomorphic functions defined in a neighborhood of the origin $(0,0) \in (\boldsymbol{C}_x)^n \times \boldsymbol{C}_y$, and

$$P(x, y, X) = X^m + \textstyle\sum_{2 \leqslant j \leqslant m} a_j(x, y) X^{m-j}.$$

Lemma. *If $a_j(x, y) \not\equiv 0$ for some $2 \leqslant j \leqslant m$, we can find positive integers $h, r \in \boldsymbol{N}$, a transformation $(x, y) \to (t, z)$ of type (NGT), and a function $g(t, z, X)$ which satisfy the following conditions:*

1) *$g(t, z, X)$ is a polynomial of degree m in X with coefficients holomorphic in (t, z) in a neighborhood of $(t, z) = (0,0)$;*

2) *$g(0,0, X) = 0$ has at least two distinct roots;*

3) *$P(x, y, X) = (t_1)^{hm} z^{rm} \times g(t, z, X/((t_1)^h z^r))$.*

Proof of lemma. Put $J = \{j \,;\, a_j(x, y) \not\equiv 0, 2 \leqslant j \leqslant m\} (\neq \emptyset)$. Denote by d_j the valuation of $a_j(x, y)$ in y. If $j \in J$, we have $d_j < \infty$ and we can write

$$a_j(x, y) = y^{d_j}(a_{j,0}(x) + y\, b_j(x, y)) \text{ with } a_{j,0}(x) \not\equiv 0.$$

For $j \in J$, we denote by α_j the valuation of $a_{j,0}(x)$ in x and by β_j the valuation of $b_j(x, y)$ in x. Put

$$\sigma = \min\{d_j/j \,;\, j \in J\} (< \infty)\,;$$
$$J^* = \{j \,;\, j \in J \text{ and } d_j/j = \sigma\} \ (\neq \emptyset)\,;$$
$$s = \min\{\alpha_j/j \,;\, j \in J^*\} (< \infty)\,;$$
$$J_0 = \{j \,;\, j \in J^* \text{ and } \alpha_j/j = s\} \ (\neq \emptyset)\,;$$
$$\mu = \max\{j \,;\, j \in J_0\}.$$

It is easy to see that $d_j = \sigma j$ for $j \in J^*$, $d_j > \sigma j$ for $j \in J \backslash J^*$, $\alpha_j = sj$ for $j \in J_0$, and $\alpha_j > sj$ for $j \in J^* \backslash J_0$. Since $\mu \in J_0$, $a_{\mu,0}(x)$ is expressed in the form

$$a_{\mu,0}(x) = \textstyle\sum_{|\nu| \geqslant s\mu} a_{\mu,0,\nu} x^\nu \text{ and } \sum_{|\nu| = s\mu} a_{\mu,0,\nu} x^\nu \not\equiv 0.$$

Therefore, after a linear change of variables in x we may assume that $a_{\mu,0,(s\mu,0,\ldots,0)} \neq 0$.

Let us choose $p \in \boldsymbol{N}^*$ so that the following conditions are satisfied:

1) $\sigma p \in \boldsymbol{N}^*$;
2) for $j \in J^*$, $p \geqslant k(sj - \beta_j)$;
3) for $j \in J \backslash J^*$, $p \geqslant k(sj - \alpha_j)/(d_j - \sigma j)$;
4) for $j \in J \backslash J^*$, $p \geqslant k(sj - \beta_j)/(d_j - \sigma j + 1)$.

Choose also $q \in \boldsymbol{N}^*$ so that $\sigma q \in \boldsymbol{N}^*$, and $k \in \boldsymbol{N}^*$ so that $sk \in \boldsymbol{N}^*$. By using these p, q, k, we first transform $x \to t$ by

$$x_1 = (t_1)^k,\ x_2 = (t_1)^k t_2, \ldots, x_n = (t_1)^k t_n$$

and then we put $y = (t_1)^p z^q$. Denote by $A_j(t, z)$ the transform of $a_j(x, y)$ for $j \in J$; then $A_j(t, z)$ is expressed in the form

$$A_j(t, z) = z^{qd_j}(A_{j,0}(t) + z^q B_j(t, z)) \text{ with } A_{j,0}(t) \not\equiv 0.$$

It is easy to see that the valuation of $A_{j,0}(t)$ in t is equal to or greater than $k\alpha_j + pd_j$ and the valuation of $B_j(t, z)$ in t is equal to or greater than $k\beta_j + pd_j + p$. If we put $h = \sigma p + sk$ and $r = \sigma q$, we can see:

i) if $j \in J_0$ we have $qd_j = rj$, $k\alpha_j + pd_j = hj$ and $k\beta_j + pd_j + p \geqslant hj$;
ii) if $j \in J^* \backslash J_0$ we have $qd_j = rj$, $k\alpha_j + pd_j > hj$ and $k\beta_j + pd_j + p \geqslant hj$;
iii) if $j \in J \backslash J^*$ we have $qd_j > rj$, $k\alpha_j + pd_j \geqslant hj$ and $k\beta_j + pd_j + p \geqslant hj$.

Moreover we have $(A_{\mu,0}(t)/(t_1)^{h\mu})|_{t=0} = a_{\mu,0,(s\mu,0,\ldots,0)} \neq 0$.

Now, let us define $g(t, z, X)$ by the follow-

ing:
$g(t, z, X) = X^m + \sum_{j\in J}(A_j(t, z)/((t_1)^{hj}z^{rj}))X^{m-j}$.
Then by i), ii) and iii) we see that $g(t, z, X)$ satisfies the conditions 1) and 3) in the lemma. Note that $g(0,0, X)$ is expressed in the form

$$g(0,0, X) = X^m + \sum_{j\in J_0} C_j X^{m-j}$$

for some $C_j \in \boldsymbol{C}$ $(j \in J_0)$ and that $C_\mu = a_{\mu,0,(s\mu,0,\dots,0)} \neq 0$. Since $\mu = \max\{j\,; j \in J_0\}$ and $J_0 \subset \{2, \dots, m\}$ hold, we can easily see the condition 2) in the lemma.

§3. An application. Let $n = 1$, $x \in \boldsymbol{C}$, $y \in \boldsymbol{C}$, $\theta = x(d/dx)$, and let us consider the following ordinary differential equation:

(e) $(\theta y)^m + \sum_{1\leqslant j\leqslant m} a_j(x, y)y^j(\theta y)^{m-j} = 0$,

where $a_j(x, y)$ $(1 \leqslant j \leqslant m)$ are holomorphic functions defined in a neighborhood of the origin (0,0) of $\boldsymbol{C}_x \times \boldsymbol{C}_y$. In (e), $y = y(x)$ is regarded as an unknown function. By applying Theorem 2 we get

Proposition. *By a transformation* $x = t^k$ *and* $y = t^p z^q$ *for some* $k, p, q \in \boldsymbol{N}^*$, *the equation* (e) *is reduced in a neighborhood of the origin* $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_z$ *into* m *equations of the following form*:

(3.1) $t(dz/dt) = \varphi_j(t, z)z, \quad (1 \leqslant j \leqslant m)$,

where $\varphi_j(t, z)$ $(1 \leqslant j \leqslant m)$ *are holomorphic functions defined in a neighborhood of* $(0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_z$.

In general, an equation of the form $t(dy/dt) = f(t, y)$ is called the Briot-Bouquet equation if $f(0,0) = 0$ is satisfied. If this equation has a holomorphic solution, we can reduce this into an equation of the form $t(dz/dt) = \varphi(t, z)z$.

Thus, the equation (3.1) is a particular case of the Briot-Bouquet equation and we already know many results on the equation (3.1) (for example, see [2], [3]).

Proof of proposition. Let us write the equation (e) in the form

(e_1) $(\theta y/y)^m + \sum_{1\leqslant j\leqslant m} a_j(x, y)(\theta y/y)^{m-j} = 0$,

and let us apply Theorem 2 to this form. Since we are considering the case $n = 1$, a composition of a finite number of transformations of type (NGT) is also written as $x = t^k$, $y = t^p z^q$ for some $k, p, q \in \boldsymbol{N}^*$. Then

$$(\theta y/y) = (qt(dz/dt) + pz)/(kz)$$

and therefore by Theorem 2 (e_1) is decomposed into

$$\Pi_{1\leqslant j\leqslant m}(((qt(dz/dt) + pz)/(kz)) - \phi_j(t, z)) = 0.$$

This implies that (e) is reduced to

$$t(dz/dt) = (-(p/q) + (k/q)\phi_j(t, z))z, \quad (1 \leqslant j \leqslant m).$$

References

[1] R. Gérard and H. Tahara: On the factorization of an equation of the form $F(x, y, \tau y) = 0$ where τ is a meromorphic vector field. Structure of solutions of differential equations, Katata/Kyoto, 1995, World Scientific, 151–167 (1996).

[2] R. Gérard and H. Tahara: Singular nonlinear partial differential equations. Vieweg (1996).

[3] M. Hukuhara, T. Kimura, and T. Matuda: Equations différentielles ordinaires du premier ordre dans le champ complexe. Publ. of Math. Soc. Japan (1961).