Best Constant in Weighted Sobolev Inequality*)

By Toshio HORIUCHI

Department of Mathematical Science, Ibaraki University

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1. Introduction and results. The purpose of the present paper is to study the best constant in the imbedding theorems for the weighted Sobolev spaces with weight functions being powers of $|x|$. We shall deal with the weighted Sobolev spaces denoted by $W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$ and $R^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$, where p, n, α, β satisfy $n \geq 2$, $1 < p < n/(1-\alpha+\beta)$ and $\alpha, \beta > -n/p$ (See also (1.5)). Let $L^p_\alpha(\boldsymbol{R}^n)$ denote the space of Lebesgue measurable functions, defined on $\boldsymbol{R}^n$, for which

$$\text{(1.1)} \quad \| u\,; L^p_\alpha(\boldsymbol{R}^n) \| = \left(\int_{\boldsymbol{R}^n} |u|^p |x|^{\alpha p}\, dx \right)^{1/p} < +\infty.$$

$W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$ is defined as the completion of $C^\infty_0(\boldsymbol{R}^n)$ with respect to the norm

$$\text{(1.2)} \quad \| u\,; W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n) \| = \| u\,; L^q_\beta(\boldsymbol{R}^n) \| + \| |\nabla u|\,; L^p_\alpha(\boldsymbol{R}^n) \|,$$

where $q = q(p, \alpha, \beta, n)$ is the so-called Sobolev exponent defined by

$$\text{(1.3)} \quad q = q(p, \alpha, \beta, n) \equiv \frac{np}{n - p(1-\alpha+\beta)}.$$

Here we note that q satisfies the equality in (1.5), and if $\alpha = \beta$ then q equals $np/(n-p)$, $R^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$ is defined as

$$\text{(1.4)} \quad R^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n) = \{u \in W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)\,; u \text{ is a radial function}\}.$$

We shall study the following variational problems. Assume that p, q, n, α and β satisfy

$$\text{(1.5)} \quad \begin{cases} 1 < p < +\infty,\ (1-\alpha+\beta)p < n, & n \geq 2, \\ 0 < 1/p - 1/q = (1-\alpha+\beta)/n \end{cases}$$

and

$$\text{(1.6)} \quad -n/q < \beta \leq \alpha.$$

Under these assumptions (1.5) and (1.6), we set

$$(P) \quad S(p, q, \alpha, \beta, n) = \inf\left[\int_{\boldsymbol{R}^n} |\nabla u|^p |x|^{p\alpha}\, dx\,; \ u \in W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n),\ \| u\,; L^q_\beta(\boldsymbol{R}^n) \| = 1 \right].$$

In the following problem (P_R), we assume instead of the inequality (1.6)

$$\text{(1.7)} \quad -n/q < \beta.$$

Under the assumptions (1.5) and (1.7), we set

$$(P_R) \quad S_R(p, q, \alpha, \beta, n) = \inf\left[\int_{\boldsymbol{R}^n} |\nabla u|^p |x|^{p\alpha}\, dx\,; \ u \in R^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n),\ \| u\,; L^q_\beta(\boldsymbol{R}^n) \| = 1 \right].$$

By a suitable change of variables this variational problem (P_R) in the radial space $R^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$ is reduced to prove the classical Sobolev inequality, which was solved by G. Talenti using the notion of Hilbert invariant integral (Lemma 2 in [12]), and the infimum is achieved by functions of the form

$$\text{(1.8)} \quad v(x) = [a + b|x|^{\frac{hp}{p-1}}]^{1-\frac{n}{p(1-\alpha+\beta)}}, \quad h = \frac{(1-\alpha+\beta)(n-p+p\alpha)}{n - p(1-\alpha+\beta)}.$$

Then with somewhat more calculations we see

Lemma 1.1. *Assume that* (1.5) *and* (1.7). *Then we have*

$$\text{(1.9)} \quad S_R(p, q, \alpha, \beta, n) = I_R(p, q, \alpha, \beta, n), \text{ where}$$

$$\text{(1.10)} \quad I_R(p, q, \alpha, \beta, n) = \pi^{\frac{p\gamma}{2}} \cdot n \cdot \left(\frac{n-\gamma p}{p-1}\right)^{p-1} \cdot \left(\frac{n-p+p\alpha}{n-\gamma p}\right)^{p-\frac{p\gamma}{n}} \cdot \left(\frac{2(p-1)}{\gamma p}\right)^{\frac{p\gamma}{n}} \times \left\{\frac{\Gamma(n/\gamma p)\Gamma(n(p-1)/\gamma p)}{\Gamma(n/2)\Gamma(n/\gamma)}\right\}^{\frac{p\gamma}{n}},$$

where $\gamma = 1 - \alpha + \beta$. *In particular if* $\alpha = \beta$, *then we have*

$$\text{(1.11)} \quad S_R(p, q, \alpha, \alpha, n) = S(p, q, n) \cdot \left(\frac{n-p+p\alpha}{n-p}\right)^{p-\frac{p}{n}},$$

where we set $S(p, q, n) = S(p, q, 0, 0, n)$ *conventionally.*

Therefore we immediately get

Lemma 1.2. *Assume that* $1/p - 1/q = 1/n$, $1 < p < n$ *and* $n > 2$. *If* $\alpha > 0$ [*respectively* $\alpha < 0$], *then it holds that* $S(p, q, n) < S_R(p, q, \alpha, \alpha, n)$ [*respectively* $S(p, q, n) > S_R(p, q, \alpha, \alpha, n)$]. *Here* $S(p, q, n) = S(p, q, 0, 0, n)$ *as in* (1.11).

From this lemma it seems that if $\alpha \leq 0$, $S_R(p, q, \alpha, \beta, n)$ is also the best constant for

*) Dedicated to Professor S. Mizohata on his Seventieth Birthday.

the problem (P), and in the subsequent argument this proves to be true.

Lemma 1.3. *Assume that* p, q, α, β, n *satisfy* (1.5) *and* (1.6). *Then we have the followings.*

(1) *If* $\beta \leq \alpha \leq 0$, *then*

$$S(p, q, \alpha, \beta, n) = S_R(p, q, \alpha, \beta, n) = I_R(p, q, \alpha, \beta, n).$$

(2) *Assume that* (1.7) *instead of the inequality* (1.6). *If* $\alpha \leq 0$ *and* $\beta \geq 0$, *then*

$$S_R(p, q, \alpha, \beta, n) = I_R(p, q, \alpha, \beta, n).$$

Proof. The proof is done by the use of the spherically symmetric decreasing rearrangement and Lemma 1.1 and Lemma 1.2.

Now we are in a position to state our main result.

Theorem 1.4. (1) *Assume that* $0 < \alpha = \beta$, $1/2 - 1/q = 1/n$, $n > 2$. *Then it holds that*

$$(1.12) \quad S(2, q, \alpha, \alpha, n) = S(2, q, 0, 0, n) = S(2, q, n).$$

Moreover there exists no extremal function which attains the infimum in $W^{1,p}_{\alpha,\alpha}(\boldsymbol{R}^n)$.

(2) *Assume that* $\alpha > 0, \alpha > \beta, 0 < 1/p - 1/q = (1 - \alpha + \beta)/n, n \geq 2$ *and* $1 < p < \dfrac{n}{1-\alpha+\beta}$. *Then the infimum* $S(p, q, \alpha, \beta, n)$ *is attained by an extremal function* u *in* $W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$ *and this* u *satisfies in distribution sense the equation:*

$$(1.13) \quad -\operatorname{div}(|x|^{p\alpha}|\nabla u|^{p-2}\nabla u) = S(p, q, \alpha, \beta, n) \cdot |x|^{\beta q}|u|^{q-2}u.$$

Remark 1. See [7], for the detailed proof of the lemmas, Theorem 1.4 and the related informations.

Remark 2. In the assertion (1), the best constant $S(p, q, \alpha, \alpha, n)$ is not known unless $p = 2$. Because the proof in this paper essentially depends on the linearlity of the Euler Lagrange equation. But at least we see that $S(p, q, \alpha, \alpha, n) \leq S(p, q, n)$ in the proof of the assertion (1). Though the best constant in assertion (2) is also unkown in general, we can show the following by the method of Lagrange multiplier.

Proposition 1.5. *Assume that*

$$(1.14) \quad 2\alpha = \beta(\alpha)q(\alpha), \quad q(\alpha) = \frac{2(n+2\alpha)}{n+2\alpha-2}, \quad \beta(\alpha) = \frac{n+2\alpha-2}{n+2\alpha}\alpha.$$

In addition we assume that 2α *is a positive integer. Then it holds that*

$$(1.15) \quad S(2, q(\alpha), \alpha, \beta(\alpha), n) = I_R(2, q(\alpha), \alpha, \beta(\alpha), n) = S(2, 2n/(n-2), n+2\alpha) \cdot \pi^{-2\alpha/(n+2\alpha)} \cdot \left(\frac{\Gamma((n+2\alpha)/2)}{\Gamma(n/2)}\right)^{2/(n+2\alpha)}.$$

We also note that if we replace the weight function $|x|$ by $|x_n|$, we can show a similar result.

2. A sketch of the proof of Theorem 1.4. For a nonnegative function $f \in C^0_0(\boldsymbol{R}^n)$, we denote by $\boldsymbol{S}(f)$ the spherically symmetric decreasing rearrangement of f (the Schwarz symmetrization of f). That is:

$$(2.1) \quad \boldsymbol{S}(f)(x) = \sup\{t\,; \mu(t) > |S^{n-1}| \cdot |x|^n\}, \quad \mu(t) = |\{x\,; f(x) > t\}|.$$

Lemma 2.1. *Let* $\boldsymbol{S}(f)$ *be the spherically symmetric decreasing rearrangement of a nonnegative function* $f \in C^0_0(\boldsymbol{R}^n)$. *Let* $g \in C^0((0, \infty))$ *be a nonnegative decreasing function. Then, for every exponent* $p \geq 1$, *the followings hold:*

$$(2.2) \quad \int_{\boldsymbol{R}^n} \boldsymbol{S}(f)^p\,dx = \int_{\boldsymbol{R}^n} f^p\,dx, \quad \int_{\boldsymbol{R}^n} |\nabla \boldsymbol{S}(f)|^p\,dx \leq \int_{\boldsymbol{R}^n} |\nabla f|^p\,dx$$

$$(2.3) \quad \int_{\boldsymbol{R}^n} \boldsymbol{S}(f)^p\, g(|x|)dx \geq \int_{\boldsymbol{R}^n} f^p \cdot g(|x|)dx.$$

The next one is a variant of the Hardy-Sobolev inequality.

Lemma 2.2. *Assume that* $f \in C^2(\Omega)$, $u \in C^\infty_0(\Omega)$, $\Omega \subset \boldsymbol{R}^n$ $(n > 2)$. *Let us set* $v(x) = \boldsymbol{S}(|f \cdot u|)(x)$. *Then it holds that*

$$(2.4) \quad \int_{\boldsymbol{R}^n} |\nabla v|^2\,dx + \frac{1}{2}\int_\Omega u^2[\Delta(f^2) - 2|\nabla f|^2]dx \leq \int_\Omega |\nabla u|^2 f^2\,dx.$$

Proof of the Assertion 1. By the use of Lemma 2.2 for $f = |x|^\alpha$, $\Omega = \boldsymbol{R}^n$ and Sobolev inequality without weights, we see that

$$(2.5) \quad S(2, q, n)\left(\int_{\boldsymbol{R}^n} |v|^q\,dx\right)^{2/p} + \alpha(\alpha+n-2)\int_{\boldsymbol{R}^n} u^2|x|^{2(\alpha-1)}dx \leq \int_{\boldsymbol{R}^n} |\nabla u|^2|x|^{2\alpha}\,dx.$$

The rest of the proof is now obvious.

Proof of the Assertion 2. Let us set for $u \in W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$

$$(2.6) \quad \begin{cases} J(u) = \displaystyle\int_{\boldsymbol{R}^n} |u|^q|x|^{\beta q}\,dx, \\ E(u) = \displaystyle\int_{\boldsymbol{R}^n} |\nabla u|^p|x|^{p\alpha}\,dx, \\ S^\lambda = \inf[E(u)\,; J(u) = \lambda, u \in W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)], \quad 0 < \lambda \leq 1. \end{cases}$$

Assume that $\{u_j\} \subset W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$ is a minimizing sequence such that

(2.7) $\lim_{j\to+\infty} E(u_j) = S \equiv S(p, q, \alpha, \beta, n)$,

$$J(u_j) = 1 \quad (j = 1,2,3,\ldots).$$

In order to prove the existence of the extremal function in $W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$, first we show the tightness of the sequence considered. Let us also set

(2.8) $\rho_j = |\nabla u_j|^p |x|^{\alpha p} + |u_j|^q |x|^{\beta q}$,

$$Q_j(R) = \int_{B_R(0)} \rho_j \, dx \quad (j = 1,2,3,\ldots).$$

By the homogeneity we may assume from the first

(2.9) $Q_j(1) = \frac{1}{2}, \quad j = 1,2,3,\ldots$

Then we see

Key lemma. *For an arbitrary $\varepsilon > 0$, there exists some positive number R such that we have*

(2.10) $\int_{\boldsymbol{R}^n\setminus B_R(0)} \rho_j \, dx < \varepsilon, \quad (j = 1,2,3,\ldots)$

After all we see that under the condition (2.9) the minimizing sequence $\{u_j\}_{j=1}^\infty \subset W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n)$ and $\{\rho_j\}_{j=1}^\infty$ are tight in $L^q_\beta(\boldsymbol{R}^n)$ and a space of all bounded measures on $\boldsymbol{R}^n$respectively. To see the existence of extremals, we have only to apply an apparent variant of the concentration compactness lemma due to P. Lions in [8] and [9]. For the complete proof see [7].

Appendix (Imbedding theorems). For the sake of self-containedness we briefly describe the imbedding theorems for the weighted Sobolev spaces, which fulfil fundamental role in the argument of this paper. $H^{1,p}_\alpha(\boldsymbol{R}^n)$ is defined as the completion of $C^\infty_0(\boldsymbol{R}^n)$ with respect to

(A.1) $\| u ; H^{1,p}_\alpha(\boldsymbol{R}^n) \| = \| u ; L^p_\alpha(\boldsymbol{R}^n) \| + \| |\nabla u| ; L^p_\alpha(\boldsymbol{R}^n) \|.$

Imbedding Theorem A.1. *Let p satisfy $1 \leq p < +\infty$ and let n satisfy $n \geq 2$. Let D be a bounded subdomain of $\boldsymbol{R}^n$. Then the following imbeddings are valid:*

Case A Suppose $(1 - \alpha + \beta)p < n$, $0 \leq 1/p - 1/r \leq (1 - \alpha + \beta)/n$ and $-n/r < \beta \leq \alpha$, then

(A.2) $H^{1,p}_\alpha(\boldsymbol{R}^n) \to L^r_\beta(\boldsymbol{R}^n)$,

$$p \leq r \leq np/[n - p(1 - \alpha + \beta)].$$

Case B Suppose $(1 - \alpha + \beta)p = n$ and $0 \leq \beta \leq \alpha$, then

(A.3) $H^{1,p}_\alpha(\boldsymbol{R}^n) \to L^r_\beta(\boldsymbol{R}^n), \quad p \leq r < +\infty.$

Case C Suppose $n < (1 - \alpha + \beta)p$ and $0 \leq \beta \leq \alpha$, then

(A.4) $H^{1,p}_\alpha(\boldsymbol{R}^n) \to C^{0,\lambda}_\beta(\boldsymbol{R}^n)$,

$$0 \leq \lambda \leq 1 - \alpha + \beta - n/p.$$

Moreover if $0 \leq 1/p - 1/r < (1 - \alpha + \beta)/n$, then the following restrictions of the mapping defined by (A.5) *are compact;*

(A.5) $H^{1,p}_\alpha(\boldsymbol{R}^n) \to L^r_\beta(D)$,

$$p \leq r < np/[n - p(1 - \alpha + \beta)].$$

From the assertion in the case A, we see that $W^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n) \supset H^{1,p}_\alpha(\boldsymbol{R}^n)$. The proof of this theorem is seen in many places, for instance, (A.2) is seen in Maz'ja's book [10; Thorem 1 and its corollaries in §2]. We note that these are also obtained as a corollary to the more general imbedding theorem in the author's paper [6; Theorem 1 in §3]. If we restrict ourselves in last statement of Theorem to consider radial functions, then we have the following result.

Proposition A.2. *Let $B(0)$ be a ball with a center being 0 in R^n. If $(1 - \alpha + \beta)p < n$, $0 \leq 1/p - 1/r < (1 - \alpha + \beta)/n$ and $\beta > -n/r$, then the following imbedding mappings are compact:*

(A.5) $R^{1,p}_{\alpha,\beta}(\boldsymbol{R}^n) \to L^r_\beta(B(0))$,

$$p \leq r < np/[n - p(1 - \alpha + \beta)].$$

In this proposition, r may exceed the so-called Sobolev exponent $np/(n - p)$ provided $\beta > \alpha$, because elements in $R^{1,p}_{\alpha,\beta}(D)$ are essentially depend upon one variable. And the proof is elementary by the use of the polar coordinate system. For the details see [4; Lemma 10] for instance.

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