Borel-Weil Type Theorem for the Flag Manifold of a Generalized Kac-Moody Algebra

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In this article, we construct and analyze a certain manifold $\boldsymbol{X}$ associated with a generalized Kac-Moody (GKM) algebra. In the case of a Kac-Moody algebra, $\boldsymbol{X}$ equals the flag manifold constructed in [3]. So we call this manifold the flag manifold of the GKM algebra.

We also give certain kinds of line bundles on the flag manifold $\boldsymbol{X}$, and determine the spaces of global sections of the bundles. The author emphasizes that the analysis of such spaces plays an important role in the highest weight representation theory in the case of a Kac-Moody algebra, especially in the proof of the Kazhdan-Lusztig conjecture.

§1. A generalized Kac-Moody algebra. Let n be a positive integer, I an n-elements set, and $A = (a_{ij})_{i,j\in I}$ a real matrix indexed by I satisfying (1) $a_{ii} = 2$ or ≤ 0, (2) $a_{ij} \leq 0$ if $i \neq j$, and in addition $a_{ij} \in \boldsymbol{Z}$ if $a_{ii} = 2$, and (3) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$. In this paper, we further assume that A is symmetrizable.

Let $(\mathfrak{h}_{\boldsymbol{R}}, \Pi, \check{\Pi})$ be a realization of A over $\boldsymbol{R}$. We denote by $\mathfrak{g}_{\boldsymbol{R}} = \mathfrak{g}_{\boldsymbol{R}}(A)$ the GKM algebra over $\boldsymbol{R}$ constructed from this realization and the Chevalley generators $e_i, f_i (i \in I)$. Then $\mathfrak{g} = \mathfrak{g}(A) \overset{\text{def}}{=} \boldsymbol{C} \otimes_{\boldsymbol{R}} \mathfrak{g}_{\boldsymbol{R}}$ and $\mathfrak{h} = \boldsymbol{C} \otimes_{\boldsymbol{R}} \mathfrak{h}_{\boldsymbol{R}}$ are the complex GKM algebra with Cartan matrix A and its Cartan subalgebra, respectively. We use the following standard notations:

- $Q_+ = \sum_{i\in I}\boldsymbol{Z}_{\geq 0}\alpha_i$, and $\lambda \geq \lambda' \overset{\text{def}}{\Leftrightarrow} \lambda - \lambda' \in Q_+$,
- $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}\,;\, [h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}\}$ is the root space for $\alpha \in \mathfrak{h}^*$,
- $\Delta = \{a \in \mathfrak{h}^* \setminus \{0\}\,;\, \mathfrak{g}_\alpha \neq 0\}$ is the set of roots,
- $\Delta_+ = \Delta \cap Q_+$ is the set of positive roots, and $\mathfrak{n}_\pm = \sum_{\alpha\in\Delta_+}\mathfrak{g}_{\pm\alpha}$,
- $I^{re} = \{i \in I\,;\, a_{ii} = 2\}$,
- $\Pi^{re} = \{\alpha_i\,;\, i \in I^{re}\}$ is the set of real simple roots,
- $\check{\Pi}^{re} = \{\check{\alpha}_i\,;\, i \in I^{re}\}$ is the set of real simple coroots,
- $r_i : \mathfrak{h}^* \ni \lambda \mapsto \lambda - \lambda(\check{\alpha}_i)\alpha_i \in \mathfrak{h}^* (i \in I^{re})$ are the simple reflections,
- $W = \langle r_i\,;\, i \in I^{re}\rangle \subset \mathrm{GL}(\mathfrak{h}^*)$ is the Weyl group,
- $P = \{\lambda \in \mathfrak{h}^*_{\boldsymbol{R}}\,;\, \lambda(\check{\alpha}_i) \in \boldsymbol{Z}$ for all $i \in I^{re}\}$ is the set of integral weights,
- $P_+ = \{\lambda \in P\,;\, \lambda(\check{\alpha}_i) \geq 0$ for all $i \in I\}$ is the set of dominant integral weights,
- $P_+(I^{re}) = \{\lambda \in P\,;\, \lambda(\check{\alpha}_i) \geq 0$ for all $i \in I^{re}\}$,
- $P_{++} = \{\lambda \in P\,;\, \lambda(\check{\alpha}_i) > 0$ for all $i \in I\}$ is the set of regular dominant integral weights.

We denote by $L(\lambda)$ the irreducible $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$.

Let V be a $\mathfrak{g}$-module. If V is expressed as the direct sum $\sum_{\mu\in\mathfrak{h}^*} V_\mu$ of the subspaces $V_\mu \overset{\text{def}}{=} \{v \in V\,;\, hv = \mu(h)v$ for all $h \in \mathfrak{h}\}$, then we call the module V $\mathfrak{h}$-diagonalizable, each subspace V_μ the weight space of weight μ, and each element of the set $P(V) \overset{\text{def}}{=} \{\mu \in \mathfrak{h}^*\,;\, V_\mu \neq 0\}$ a weight of V. If in addition e_i and f_i $(i \in I^{re})$ act locally nilpotently, V is called integrable.

Let V be an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module with finite-dimensional weight spaces. We denote by V^* the $\mathfrak{g}$-invariant subspace $\sum_{\mu\in P(V)}(V_\mu)^*$ of the $\mathfrak{g}$-module $\mathrm{Hom}_{\boldsymbol{C}}(V, \boldsymbol{C})$. We have $(V^*)_\mu = (V_{-\mu})^*$ for any $\mu \in \mathfrak{h}^*$.

§2. Completions of algebras and modules. Let $w \in W$, and put

$$\Delta_+(w) = \Delta_+ \cap w\Delta_+,\ \mathfrak{n}_\pm(w) = \textstyle\sum_{\alpha\in\Delta_+(w)}\mathfrak{g}_{\pm\alpha},$$
$$\hat{\mathfrak{n}}_\pm(w) = \textstyle\prod_{\alpha\in\Delta_+(w)}\mathfrak{g}_{\pm\alpha},\ {}^w\mathfrak{n}_\pm = \textstyle\sum_{\alpha\in w\Delta_+}\mathfrak{g}_{\pm\alpha},$$
$${}^w\hat{\mathfrak{n}}_\pm = \textstyle\prod_{\alpha\in w\Delta_+}\mathfrak{g}_{\pm\alpha},\ \hat{\mathfrak{g}} = {}^w\hat{\mathfrak{n}}_- + \mathfrak{h} + {}^w\mathfrak{n}_+.$$

Clearly, $\mathfrak{n}^w_\pm$, $\mathfrak{n}_\pm(w)$, and ${}^w\mathfrak{n}_\pm$ are subalgebras of $\mathfrak{g}$. Furthermore, the bracket products in ${}^w\mathfrak{n}_\pm$ (resp. $\mathfrak{g}$) are naturally extended to ${}^w\hat{\mathfrak{n}}_\pm$ (resp. $\hat{\mathfrak{g}}$), and the definition of the Lie algebra $\hat{\mathfrak{g}}$ is independent of w.

Let V be an integrable $\mathfrak{g}$-module such that $P(V) \subset (\lambda_1 - Q_+) \cup \ldots \cup (\lambda_m - Q_+)$ for some $\lambda_1, \ldots, \lambda_m \in \mathfrak{h}^*$. We consider the direct product of weight spaces of V :

$$\hat{V}=\prod_{\mu\in P(V)} V_\mu.$$

Then the action of $\hat{\mathfrak{g}}$ on $\hat{V}$ naturally arises from the action of $\mathfrak{g}$ on V.

On direct product spaces, we always consider direct product topologies. Then all the operations introduced above are continuous. Moreover, all the topological Lie algebras ${}^w\hat{\mathfrak{n}}_\pm$ are isomorphic to $\hat{\mathfrak{n}}_\pm := {}^e\hat{\mathfrak{n}}_\pm$, where e is the unity of the Weyl group W.

Let $\lambda, \lambda' \in P_+(I^{\mathrm{re}})$. We denote by $\hat{L}(\lambda)\otimes\hat{L}(\lambda')$ the topological tensor product of $\hat{L}(\lambda) := \widehat{L(\lambda)}$ and $\hat{L}(\lambda') := \widehat{L(\lambda')}$:

$$\hat{L}(\lambda)\otimes\hat{L}(\lambda') \overset{\mathrm{def}}{=} \prod_{\alpha\in Q_+} \sum_{\substack{\beta,\beta'\in Q_+\\ \beta+\beta'=\alpha}} L(\lambda)_{\lambda-\beta}\otimes L(\lambda')_{\lambda'-\beta'}.$$

If $\lambda, \lambda' \in P_+$, then it is clear that $\hat{L}(\lambda)\otimes\hat{L}(\lambda')$ is decomposed into $\hat{\mathfrak{g}}$-submodules of the form $\hat{L}(\mu')$ with $\mu' \in \lambda+\lambda'-Q_+$, and that $\hat{L}(\lambda+\lambda')$ appears as a direct summand with multiplicity 1.

By making use of the Campbell-Hausdorff formula, we can give an algebraic group structure to the set $\exp{}^w\hat{\mathfrak{n}}_-$, which consists of symbols $\exp x$ ($x\in{}^w\hat{\mathfrak{n}}_-$). Furthermore, we define the actions of $\exp{}^w\hat{\mathfrak{n}}_-$ on $\hat{V}$ and on $\hat{\mathfrak{g}}$ by

$$(\exp x)v \overset{\mathrm{def}}{=} \sum_{j\geq 0}\frac{1}{j!}x^j v,$$

$$\mathrm{Ad}(\exp x)y \overset{\mathrm{def}}{=} \sum_{j\geq 0}\frac{1}{j!}(\mathrm{ad}\,x)^j y,$$

for $x\in{}^w\hat{\mathfrak{n}}_-$, $v\in\hat{V}$, and $y\in\hat{\mathfrak{g}}$.

§3. Construction of the flag manifold. Let $\Lambda_1,\ldots,\Lambda_m$ be a sequence of regular dominant integral weights. Put

$$\hat{\mathcal{V}}=\hat{\mathcal{V}}^{\Lambda_1,\ldots,\Lambda_m} \overset{\mathrm{def}}{=} \begin{cases}\{(v_1,\ldots,v_m)\in\hat{L}(\Lambda_1)^\times\times\cdots\times\hat{L}(\Lambda_m)^\times\,;\,(*)\} & (m\geq 2),\\ \{v\in\hat{L}(\Lambda_1)^\times\,;\,(v,v)\in\hat{\mathcal{V}}^{\Lambda_1,\Lambda_1}\} & (m=1),\end{cases}$$

where $\hat{L}(\Lambda_i)^\times := \hat{L}(\Lambda_i)\setminus\{0\}$, and the condition $(*)$ is given by

$$(*)\quad v_1\otimes\cdots\otimes v_m\in\hat{L}(\Lambda_1+\cdots+\Lambda_m)\subset \hat{L}(\Lambda_1)\otimes\cdots\otimes\hat{L}(\Lambda_m).$$

For each $w\in W$, define an open subset ${}^w\hat{\mathcal{V}}$ of $\hat{\mathcal{V}}$ by

$${}^w\hat{\mathcal{V}} \overset{\mathrm{def}}{=} \{(v_1,\ldots,v_m)\in\hat{\mathcal{V}}\,;\,P_{w\Lambda_i}(v_i)\neq 0 \text{ for any } i\}.$$

Then we have

Proposition 1. (1) $\hat{\mathcal{V}}=\bigcup_{w\in W}{}^w\hat{\mathcal{V}}$.
(2) *The group* $(\boldsymbol{C}^\times)^m\times\exp{}^w\hat{\mathfrak{n}}_-$ *acts on* ${}^w\hat{\mathcal{V}}$ *simply transitively for any* $w\in W$.

We consider the complex manifold structure on $\hat{\mathcal{V}}$ such that each open subset ${}^w\hat{\mathcal{V}}$ is diffeomorphic to $(\boldsymbol{C}^\times)^m\times\hat{\mathfrak{n}}_-$ under (2) above, and such that the open covering $\hat{\mathcal{V}}=\bigcup_{w\in W}{}^w\hat{\mathcal{V}}$ is an atlas. Put $\boldsymbol{X}\overset{\mathrm{def}}{=}\hat{\mathcal{V}}/(\boldsymbol{C}^\times)^m$, and call the manifold $\boldsymbol{X}$ the flag manifold of $\mathfrak{g}$. We see that the manifold $\boldsymbol{X}$ is independent of the choice of the sequence $\Lambda_1,\ldots,\Lambda_m$. We denote by $\hat{\pi}=\hat{\pi}^{\Lambda_1,\ldots,\Lambda_m}$ the natural projection of $\hat{\mathcal{V}}$ onto $\boldsymbol{X}$.

§4. Infinitesimal actions on the flag manifold. Although $\boldsymbol{X}$ is not defined as a homogeneous space of a group associated with $\mathfrak{g}$, we can define an infinitesimal action of $\mathfrak{g}$ on $\boldsymbol{X}$ in the following way. Let $x\in\hat{\mathfrak{g}}$ and $(v_1,\ldots,v_m)\in\hat{\mathcal{V}}$. Thanks to Proposition 1, we can find $w\in W$ and $g\in\exp{}^w\hat{\mathfrak{n}}_-$ such that $(xv_1,\ldots,xv_m)$ belongs to the space

$$(\mathrm{Ad}(g){}^w\hat{\mathfrak{n}}_-)(\boldsymbol{C}v_1\times\cdots\times\boldsymbol{C}v_m),$$

which is canonically identified with the tangent space at $(v_1,\ldots,v_m)\in\hat{\mathcal{V}}$. So the correspondence

$$\hat{x}:(v_1,\ldots,v_m)\mapsto\hat{x}_{(v_1,\ldots,v_m)}:=(xv_1,\ldots,xv_m)$$

is considered as a vector field on $\hat{\mathcal{V}}$, and the map $x\mapsto\hat{x}$ defines an infinitesimal action of $\hat{\mathfrak{g}}$ on $\hat{\mathcal{V}}$. We define a vector field $\tilde{x}$ on $\boldsymbol{X}$ by

$$\tilde{x}_{\hat{\pi}(v_1,\ldots,v_m)}\overset{\mathrm{def}}{=}\hat{\pi}_*(\hat{x}_{(v_1,\ldots,v_m)}).$$

Clearly $\tilde{x}_{\hat{\pi}(v_1,\ldots,v_m)}$ depends only on $\hat{\pi}(v_1,\ldots,v_m)$. So we obtain an infinitesimal action of $\hat{\mathfrak{g}}$ on $\boldsymbol{X}$ by $x\mapsto\tilde{x}$.

§5. Line bundles and their global sections. Let λ be an integral weight. It is easy to prove the following lemma.

Lemma 2. *There exists a sequence* $\Lambda_1,\ldots,\Lambda_m$ *of regular dominant integral weights such that* $\Lambda_1,\ldots,\Lambda_m$ *are linearly independent, and such that* $\lambda\in\boldsymbol{Z}\Lambda_1+\cdots+\boldsymbol{Z}\Lambda_m$.

Let $\Lambda_1,\ldots,\Lambda_m$ be the sequence in this lemma. Then any element of $(\boldsymbol{C}^\times)^m$ is expressed in the form $(e^{\Lambda_1(h)},\ldots,e^{\Lambda_m(h)})$ with $h\in\mathfrak{h}$. Furthermore, the map

$$(\boldsymbol{C}^\times)^m\ni(e^{\Lambda_1(h)},\ldots,e^{\Lambda_m(h)})\mapsto e^{-\lambda(h)}\in\boldsymbol{C}^\times$$

is well-defined and belongs to $\boldsymbol{C}[(\boldsymbol{C}^\times)^m]$.

For the manifold $\hat{\mathcal{V}}=\hat{\mathcal{V}}^{\Lambda_1,\ldots,\Lambda_m}$, we define an equivalence relation $\sim$ on $\hat{\mathcal{V}}\times\boldsymbol{C}$ by

$$(v_1,\ldots,v_m,c)\sim(e^{\Lambda_1(h)}v_1,\ldots,e^{\Lambda_m(h)}v_m,e^{-\lambda(h)}c)\quad\text{for some } h\in\mathfrak{h}.$$

Define $E(\lambda)=E^{\Lambda_1,\ldots,\Lambda_m}(\lambda)\overset{\mathrm{def}}{=}(\hat{\mathcal{V}}\times\boldsymbol{C})/\sim$. Clearly $E(\lambda)$ is an algebraic line bundle over $\boldsymbol{X}$, and each restriction $E(\Lambda)\,|_{{}^w\boldsymbol{X}}$ is a trivial bundle, where ${}^w\boldsymbol{X}:=\hat{\pi}({}^w\hat{\mathcal{V}})$ ($w\in W$). Moreover, we see that the definition of $E(\lambda)$ is independent of the choice of m and the sequence $\Lambda_1,\ldots,\Lambda_m$.

Let L_λ be the space of global sections s of $E(\lambda)$ which satisfy
$p \circ s \in \boldsymbol{C}[{}^w X]$ for all $w \in W$, $p \in \boldsymbol{C}[E(\lambda)\,|_{{}^w X}]$.
Then we can state our main results as follows.

Theorem 3. (1) $L_\lambda \neq 0 \Leftrightarrow \lambda \in P_+(I^{\mathrm{re}})$.
(2) *If* $\lambda \in P_+(I^{re})$, *then* $L_\lambda \simeq (U(\mathfrak{g}) \otimes_{U(\mathfrak{g}^{\mathrm{re}}+\mathfrak{u}_+^{im})} L^{\mathrm{re}}(\lambda))^*$ *as* $\mathfrak{g}$-*modules.*

Here $\mathfrak{g}^{re}$ is the Kac-Moody algeba constructed from the submatrix $(a_{ij})_{i,j\in I^{re}}$ and the realization $(\mathfrak{h}, \Pi^{\mathrm{re}}, \check{\Pi}^{\mathrm{re}})$ of $(a_{ij})_{i,j\in I^{re}}$, $\mathfrak{u}_+^{im} \overset{\mathrm{def}}{=} \sum_{\alpha\in Q_+^{im}} \mathfrak{g}_\alpha$, $Q_+^{im} \overset{\mathrm{def}}{=} Q_+ \setminus Q_+^{\mathrm{re}}$, $Q_+^{\mathrm{re}} \overset{\mathrm{def}}{=} \sum_{i\in I^{re}} \boldsymbol{Z}_{\geq 0}\alpha_i$, and $L^{\mathrm{re}}(\lambda)$ is the irreducible $\mathfrak{g}^{re}$-module with highest weight λ. We let $\mathfrak{u}_+^{im}$ act trivially on $L^{\mathrm{re}}(\lambda)$.

References

[1] R. Borcherds: Generalized Kac-Moody Algebras. J. Algebra., **115**, 501–512 (1988).

[2] V. G. Kac: Infinite Dimensional Lie Algebras. 3rd ed., Cambridge University Press, Cambridge (1990).

[3] M. Kashiwara: The flag manifold of Kac-Moody Lie algebra. Algebraic analysis, Geometry and Number theory. Johns Hopkins University Press, pp. 161–190 (1990).

[4] S. Kumar: Demazure character formula in arbitrary Kac-Moody setting. Invent. Math., **89**, 395–423 (1987).

[5] D. H. Peterson and V. G. Kac: Infinite flag varieties and conjugacy theorems. Proc. Nat. Acad. Sci. USA, **80**, 1778–1782 (1983).

[6] K. Suto: Borel-Weil type theorem for the flag manifold of a generalized Kac-Moody algebra (preprint).