Geometric Transition for a Class of Hyperbolic Operators with Double Characteristics

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1. Introduction. The Cauchy problem in the C^∞ category for hyperbolic operators with double characteristics has been studied by many authors under different geometrical perspectives. These results (see e.g. [15], [1] and the references therein) put in evidence the role of both the lower order terms and the symplectic geometry associated with the principal symbol at the double characteristic points.

The purpose of our work is the study of a case in which the set of double points Σ_2 can be written as the reunion of two nonempty subsets $\Sigma_{2,e}$, $\Sigma_{2,ne}$ i.e. the set of effectively hyperbolic double points and its Σ_2-complementary respectively. We say that in this case a "transition" occurs, or, roughly speaking, that a real eigenvalue of the fundamental matrix defined on $\Sigma_{2,e}$ vanishes on $\Sigma_{2,ne}$.

Trying to provide a unifying approach to the first of the open problems mentioned above, R. B. Melrose, [15], in 1984 formulated the following conjecture:

Conjecture. *If the Cauchy problem for P (having at most double characteristics) is well-posed in Ω_0 then $s(\rho)/\lambda(\rho)$ is uniformly bounded in $\Sigma_{2,e} \cap T^* V \setminus \{0\}$, for some neighborhood V of $0 \in \mathbf{R}^{n+1}$, where*

$$s(\rho) = |\operatorname{Im} P^s_{m-1}(\rho)|$$
$$+ \inf\{|\operatorname{Re} P^s_{m-1}(\rho) - s|; |s| \le \operatorname{Tr}^+ F_{P_m}(\rho)\}.$$

This conjecture, as it stands, has not been proved yet, however it gives a hint of what one should reasonably look for, when trying to formulate Levi conditions on the subprincipal symbol P^s_{m-1}.

The purpose of this paper is actually to understand a degenerate effectively hyperbolic case: more precisely we will show that the uniform boundedness of the ratio $s(\rho)/\lambda(\rho)$ may also serve as a sufficient condition for the well-posedness of the Cauchy problem, provided certain symplectically invariant geometric conditions are satisfied. Our Assumption (H4) is therefore a

slightly restricted reformulation of Melrose Conjecture, and this is expected since we are dealing with a sufficiency result.

Even under the usual assumption that the principal symbol P_m vanishes exactly of order two on Σ_2, one readily sees that the nature of the set $\Sigma_{2,ne}$ (henceforth denoted by Σ'_2), the order of vanishing of λ on $\Sigma_{2,ne}$ and the geometry of the transition from a symplectic point of view can be of a very wild type and may produce nontrivial situations.

Therefore we started assuming that Σ_2 is a smooth submanifold of $T^* \Omega \setminus \{0\}$. Furthermore in Section 3 below we collected a number of results serving as a symplectic classification of the transition cases that can possibly occur. In this framework Assumption (H2) isolates one of these cases.

More precisely (H2) takes a picture of $F_{P_m}(\rho)$ when ρ is inside or outside Σ'_2, avoiding for instance Jordan blocks of size 4 in the canonical form of $F_{P_m}(\rho)$, $\rho \in \Sigma'_2$ and precising that $0 \le \lambda(\rho) \le \operatorname{dist}^2_{\Sigma_2}(\rho, \Sigma'_2)$, where dist denotes any geodesic distance of $\rho \in \Sigma_2$ from Σ'_2; but this is not yet enough to prove an energy estimate eventually leading to existence and uniqueness. Indeed we recall that in the non degenerate effectively hyperbolic case (see e.g. Lemma 1.2.1 in [17]), denoting by $\Gamma_{P_m}(\rho)$ the hyperbolicity cone of the localized hyperbolic quadratic form $\sigma(X, F_{P_m}(\rho)X)$, "$P_m$ is effectively hyperbolic at ρ" if and only if "$\Gamma_{P_m}(\rho) \cap \operatorname{ran} F_{P_m}(\rho) \cap [(0, e_0)]^\sigma \ne \emptyset$, where with $V^\sigma = \{z \in T_\rho T^* \Omega \mid \sigma(z, v) = 0, \forall v \in V\}$ we define the dual with respect to the symplectic form σ of a vector subspace V of $T_\rho T^* \Omega$.

Therefore for an effectively hyperbolic operator it is always possible to find a time function f, i.e. a C^∞ function vanishing on the double set of P_m, whose Hamilton vector field $H_f(\rho)$, $\rho \in \Sigma_2$ belongs to $\Gamma_{P_m}(\rho)$ (and for which in addition

$\partial f / \partial \xi_0(\rho) = 0$ holds). The usual strategy in this setting is to take advantage of this time function, as in [15] and [17], and use a suitable weight function in order to prove an energy estimate implying well-posedness.

On the other hand one quickly realizes that (see Section 3, example B) Assumption (H2) is too static to handle degenerate effectively hyperbolic cases. The existence of a "generalized" time function adapted to a "good" factorization of P_m is what we need here in order to take care of these difficulties. Roughly speaking we need a "real" time function in $\Sigma_2 \setminus \Sigma_2'$, whose existence is of course guaranteed by (H2), such that $H_f(\rho)$, when $\rho \in \Sigma_2 \to \bar{\rho} \in \Sigma_2'$ has a smooth exit from $\Gamma_{P_m}(\rho)$ to $\ker F_{P_m}(\bar{\rho})$). The precise type of requests we are asking on the "time" function f is written down explicitly in Assumption (H3). We would also like to point out that under the geometrical prescription of Hypothesis (H2) it is always possible to exhibit a good factorization of the principal symbol of P, according to Definition 2.1 in a way reminiscent of the well-known Ivrii factorization (see e. g. Ivrii [6]). The couple (H2), (H3) selects therefore a class of differential operators with double characteristics, whose fundamental matrix admits a non-negative real eigenvalue $\lambda(\rho)$, vanishing of order at least two on Σ_2', for which the uniform boundedness of the ratio $s(\rho) / \lambda(\rho)$ in Melrose conjecture proves to be the sufficient condition ensuring the correctness of the associated Cauchy problem.

2. Assumptions and the result. Let $P(x, D)$ be a differential operator of order m,

(2.1) $P(x, D) = P_m(x, D) + P_{m-1}(x, D) + \cdots$

where $P_j(x, D)$ denotes the homogeneous part of order j of P, $j = 0, \ldots, m$. We assume that P is a differential operator with C^∞ coefficients in the open set $\Omega \subset \mathbf{R}^{n+1}$, $0 \in \Omega$. In this Section we list our assumptions on the operator P.

The first hypothesis is concerned with the intrinsic properties of the double characteristics of the principal symbol of P.

(H 1) (i) The principal symbol $P_m(x, \xi)$ of P is hyperbolic with respect to the level surfaces of x_0.

    (ii) The characteristic roots of $\xi_0 \to P_m(x, \xi_0, \xi')$ have multiplicity at most two; denoting by $\Sigma_2 = \{(x, \xi) \in T^*\Omega \setminus \{0\} \mid P_m(x, \xi) = 0, dP_m(x, \xi) = 0,$

$\partial^2 P_m(x, \xi) / \partial \xi_0^2 < 0\}$ the set of the double characteristics of $P_m(x, \xi)$, we assume that Σ_2 is a C^∞ manifold such that the canonical 1-form ω does not vanish on $T\Sigma_2$.

    (iii) $\forall \rho \in \Sigma_2$ denote by $V_{i\lambda}(\rho)$ the eigenspace corresponding to the purely imaginary eigenvalue $i\lambda$ of the fundamental matrix $F_{P_m}(\rho)$ of P_m at ρ. Then,

$$\dim \bigoplus_{\substack{i\lambda \neq 0 \\ i\lambda \in \mathrm{sp}(F_{P_m}(\rho))}} V_{i\lambda}(\rho) = \mathrm{const.}$$

    (iv) $P_m(x, \xi)$ vanishes exactly of order two on Σ_2.

The second assumption deals with the degeneracy of the real eigenvalues of $F_{P_m}(\rho)$ when ρ is close to some critical subset Σ_2' of Σ_2.

(H2) There exists a subset of Σ_2, Σ_2', such that:

    (i) $F_{P_m}(\rho)$ has a strictly positive eigenvalue, $\lambda(\rho)$ if and only if $\rho \in \Sigma_2 \setminus \Sigma_2'$.

    (ii) Let

$$r(\rho) = \mathrm{rank}\, F_{P_m}^2(\rho),$$

when $\rho \in \Sigma_2 \setminus \Sigma_2'$, and

$$r'(\rho) = \mathrm{rank}\, F_{P_m}^2(\rho),$$

when $\rho \in \Sigma_2'$. Then $r(\rho) \equiv r$, $\forall \rho \in \Sigma_2 \setminus \Sigma_2'$, $r'(\rho) \equiv r'$, $\forall \rho \in \Sigma_2'$ and moreover $r = r' + 2$.

    (iii) The function λ defined in (i) is (at least) a $C^{(1)}$ function difined on Σ_2. In particular this implies that $\lambda(\rho) \lesssim d_{\Sigma_2}^2(\rho, \Sigma_2')$ where $d_{\Sigma_2}(\cdot, \Sigma_2')$ denotes any geodesic distance in Σ_2 from the set Σ_2'.

Assumption (H2) has essentially a "static" nature, i.e. everything is looked at on a fixed point $\rho \in \Sigma_2$. We also need a more "dynamical" type of assumption, which will be stated in (H3). To this end let us define what we mean by *factorization* of the principal symbol P_m. Without loss of generality we may assume in the following that $m = 2$.

Definition 2.1. *We say that $P_2(x, \xi)$ is factorized (in the sense of Ivrii [6]) if we can find C^∞ symbols $l(x, \xi'), m(x, \xi')$ vanishing on Σ_2, homogeneous of degree one, $q(x, \xi') \geq 0$ vanishing on Σ_2 and homogeneous of degree two, such that, denoting by $\Lambda(x, \xi) = \xi_0 - l(x, \xi'), M(x, \xi) = \xi_0 - m(x, \xi')$, we have:*

    (i) $H_\Lambda(\rho) \in \ker F_{P_2}(\rho) \cap \mathrm{ran}\, F_{P_2}(\rho)$, $\forall_\rho \in \Sigma_2'$.

    (ii) $P_2(x, \xi) = -\Lambda(x, \xi)M(x, \xi) + q(x, \xi')$.

(iii) $\{\Lambda, M\}(\rho) = 0$, $\forall \rho \in \Sigma_2$.

(iv) $\forall \rho \in \Sigma_2 \setminus \Sigma_2'$ there exist U_ρ conic neighborhood of ρ, a constant, $C_\rho > 0$, such that:

$$|\{\Lambda, q\}(p')| \le C_\rho \sqrt{q}(\rho'),$$

$\rho' \in U_\rho$ and $C_\rho \le c\lambda(\rho)$ for a suitable positive constant c.

(v) $H_\Lambda(\rho) - H_M(\rho) \in \ker F_{P_2}(\rho) \cap \operatorname{ran} F_{P_2}(\rho)$, $\forall \rho \in \Sigma_2$.

(H3) Assume that $P_2(x, \xi)$ is factorized according to Definition 2.1. Then there exists a homogeneous symbol of degree zero $f(x, \xi')$ vanishing on Σ_2 such that:

(i) $\{\Lambda, f\}(\rho) \sim \lambda(\rho)$, $\forall \rho \in \Sigma_2$.

(ii) $0 \le \{\Lambda, f\}(x, \xi') \le c_1\{M, f\}(x, \xi')$, for a suitable positive constant c_1.

(iii) These exists $\kappa \in]0, 1[$ such that

$$\{q, f\}^2(x, \xi')$$
$$\le 4\kappa[\Lambda, f\}(x, \xi')\{M, f\}(x, \xi')q(x, \xi'),$$

for any $(x, \xi') \in U_{\bar\rho}$, $\bar\rho \in \Sigma_2'$, $U_{\bar\rho}$ conical neighborhood of $\bar\rho$.

The last hypothesis is concerned with the lower order terms of the operator P.

(H4) There exist two positive constants C, ε such that, $\forall \rho \in \Sigma_2$

$$\operatorname{dist}(P_1^s(\rho), ([-\operatorname{Tr}^+ F_{P_m}(\rho) + \varepsilon, \operatorname{Tr}^+ F_{P_m}(\rho) - \varepsilon] \times \{0\}) \cup \{(0,0)\})$$
$$\le C\lambda(\rho)$$

where dist denotes any distance in C.

If $\operatorname{Tr}^+ F_{P_m}(\rho) > 0$, $\forall \rho \in \Sigma_2$, then the above condition becomes:

(a) $\operatorname{dist}(P_1^s(\rho), ([-\operatorname{Tr}^+ F_{P_m}(\rho) + \varepsilon, \operatorname{Tr}^+ F_{P_m}(\rho) - \varepsilon] \times \{0\})) < C\lambda(\rho)$.

On the other hand if $\operatorname{Tr}^+ F_{P_m}(\rho) \equiv 0$ on Σ_2, then we obtain

(b) $\operatorname{dist}(P_1^s(\rho), \{0,0\}) \le C\lambda(\rho)$,

i.e. $|P_1^s(\rho)| \le C\lambda(\rho)$. In the non effectively hyperbolic case, i.e. $\lambda_{|\Sigma_2} \equiv 0$, condition (a) becomes

$$\operatorname{Im} P_1^s(\rho) = 0, \quad -\operatorname{Tr}^+ F_{P_m}(\rho) + \varepsilon \le \operatorname{Re} P_1^s(\rho)$$
$$\le \operatorname{Tr}^+ F_{P_m}(\rho) - \varepsilon,$$

while condition (b) means $P_1^s(\rho) = 0$, thus reducing to the well-known Ivrii-Petkov-Hörmander Levi conditions ([9], [4], [6]).

We are now in a position to state the main theorem:

Theorem 2.1. Let P be a second order differential operator verifying Assumptions (H1)-(H4). Then the Cauchy problem for P in Ω_t, $t \in \mathbf{R}$ is well-posed in C^∞.

3. Comments. In this Section we gather a number of results characterizing our geometrical situation as well as the different possible types of degeneracy which may occur. Due to Assumption (H1) (iv), denoting by $\bar\rho$ a point in Σ_2' and $V_{\bar\rho}$ a conical neighborhood of $\bar\rho$ we can find real valued symbols homogeneous of degree one, $\varphi_j(x, \xi')$, $j = 1, \ldots, k$, defined in $V_{\bar\rho}$ such that possibly after a conjugation with a Fourier integral operator leaving the time axis invariant P_2 becomes

$$(3.1) \quad P_2(x, \xi) = -\xi_0^2 + \sum_{j=1}^k \varphi_j^2(x, \xi'), \quad (x, \xi) \in V_{\bar\rho}.$$

The following Proposition is essentially well known and we include it for a better understanding of the examples below.

Proposition 3.1. Let P_2 be as in 3.1 above and $\rho \in \Sigma_2$. Put:

$$v_\rho = (\{\xi_0, \varphi_1\}(\rho), \cdots, \{\xi_0, \varphi_k\}(\rho)),$$

$$M_\rho = \begin{bmatrix} 0 & \{\varphi_1, \varphi_2\}(\rho) & \cdots & \{\varphi_1, \varphi_k\}(\rho) \\ \{\varphi_2, \varphi_1\}(\rho) & 0 & \cdots & \{\varphi_2, \varphi_k\}(\rho) \\ \vdots & \vdots & \ddots & \vdots \\ \{\varphi_k, \varphi_1\}(\rho) & \{\varphi_k, \varphi_2\}(\rho) & \cdots & 0 \end{bmatrix},$$

and denote by $\pi_\rho : \mathbf{R}^k \to \ker M_\rho$ the projection operator onto $\operatorname{Ker} M_\rho$. Then either

a) $\pi_0 v_\rho \ne 0$; in this case P_2 is effectively hyperbolic at $\rho \in \Sigma_2$.

or

b) $\pi_\rho v_\rho = 0$; in this case the following situations occur:

(i) $|M_\rho^{-1} v_\rho| > 1$; then $F_{P_2}(\rho)$ has two real non-zero eigenvalues.

(ii) $|M_\rho^{-1} v_\rho| = 1$; then $\ker F_{P_2}^2(\rho) \cap \operatorname{ran} F_{P_2}^2(\rho) \ne \{0\}$, i.e. $F_{P_2}(\rho)$ has a Jordan block of size 4 in its canonical form.

(iii) $|M_\rho^{-1} v_\rho| < 1$; then $\ker F_{P_2}^2(\rho) \cap \operatorname{ran} F_{P_2}^2(\rho) = \{0\}$, i.e. $F_{P_2}(\rho)$ is non effectively hyperbolic.

Here $M_\rho^{-1} v_\rho$ means the unique vector belonging to $(\ker M_\rho)^\perp$ which is mapped to v_ρ by M_ρ.

The next proposition asserts that there are no Jordan blocks of size 4 in the canonical form of $F_{P_2}(\rho)$.

Proposition 3.2. Assume that (H1), (H2) hold. Then $\forall \rho \in \Sigma_2'$,

$$\ker F_{P_2}^2(\rho) \cap \operatorname{ran} F_{P_2}^2(\rho) = \{0\}.$$

Moreover if $\rho \in \Sigma_2 \setminus \Sigma_2'$, $\pi_\rho v_\rho \ne 0$.

Proposition 3.3. Assume that (H1), (H2) hold; let $\bar\rho \in \Sigma_2'$ and $U_{\bar\rho}$ a conical neighborhood of $\bar\rho$; it is then possible to find a vector

$$\gamma(x, \xi') = (\gamma_1(x, \xi'), \ldots, \gamma_{k-\nu}(x, \xi')),$$

where $\gamma_j(x, \xi')$'s *are real symbols homogeneous of degree zero smoothly dependent on* x_0, *such that*

(i) *If*

$$\Lambda(x, \xi) = \xi_0 + \langle \gamma(x, \xi'), \varphi''(x, \xi') \rangle,$$
$$M(x, \xi) = \xi_0 - \langle \gamma(x, \xi'), \varphi''(x, \xi') \rangle,$$
$$Q(x, \xi') = | \varphi'(x, \xi') |^2 +$$
$$\langle (\mathrm{Id} - \gamma \otimes \gamma) \varphi''(x, \xi'), \varphi''(x, \xi') \rangle,$$

then

$$P_2(x, \xi) = -\Lambda M + Q ;$$

(ii) $| \gamma(\rho) | < 1$, *for every* $\rho \in \Sigma_2 \cap U_{\bar{\rho}}$;

(iii) $\{\Lambda, Q\} \leq C(Q + | \partial_0 \varphi' | \sqrt{Q})$;

(iv) $\{\Lambda, M\}(\rho) = 0$, *for every* $\rho \in \Sigma_2 \cap U_{\bar{\rho}}$.

Proposition 3.4. *Assume* (H1), (H2) *hold. Furthermore suppose that if* $\forall \bar{\rho} \in \Sigma_2'$ *there exists a closed, proper cone* $\Gamma(\bar{\rho}) \subset T_{\bar{\rho}}^* \Omega$ *such that*

$$(3.2) \qquad \lim_{\substack{\rho \to \bar{\rho} \\ \rho \in \Sigma_2 \backslash \Sigma_2'}} z_\rho \in \Gamma(\bar{\rho}),$$

where z_ρ *denotes the eigenvector corresponding to the positive eigenvalue* $\lambda(\rho)$, $\rho \in \Sigma_2 \backslash \Sigma_2'$, *then condition* (i) *in Assumption* (H3) *can always be fulfilled.*

Remark 3.1. *If the manifold* Σ_2' *is cylindrical with respect to* x_0, *then* $\Lambda = M = \xi_0$ *yields a factorization in the sense of Definition 2.1, and condition* (iii) *of* (H3) *becomes:*

$$\{\xi_0, f\}^2 (x, \xi) P_2(x, \xi) + C\{P_2, f\}^2(x, \xi) \geq 0,$$

for a suitable positive constant C, *in a neighborhood of a point* $\bar{\rho} \in \Sigma_2'$.

We now exhibit a number of examples and counterexamples to our Assumptions (H1)-(H4). Let us start first with some models which do not satisfy at least one of the above mentioned assumptions.

A. Let $P_2(x, \xi) = -\xi_0^2 + (x_0 x_2^2 - x_1)^2 \xi_n^2 + r\xi_1^2$, near $(0, e_n)$, $r > 0$. Using the notations of Proposition 3.1 we have

$$v = (x_2^2, 0) \quad \text{and} \quad M = \begin{bmatrix} 0 & \sqrt{r} \\ -\sqrt{r} & 0 \end{bmatrix},$$

hence $\pi v = 0$ and $M^{-1} v = \left(0, \dfrac{x_2^2}{\sqrt{r}}\right)$ which can coincide with one of the cases of item b) in Proposition 3.1 according to $x_2^2 < \sqrt{r}$, $x_2^2 = \sqrt{r}$ or $x_2^2 > \sqrt{r}$. Therefore Assumption (H2) is *not* satisfied.

B. Let

$$P_2(x, \xi) = -\xi_0^2 + (x_0(x_2^2 - x_3^2) - x_1)^2 \xi_n^2 + (2bx_2x_3x_0 - x_4)^2 \xi_n^2,$$

near $(0, e_n)$, $b \in \mathbf{R}$. It is easy to show that Hypotheses (H1), (H2) are verified and that $\lambda \sim x_2^2 + x_3^2$. Moreover $\Lambda(x, \xi) = M(x, \xi) = \xi_0$ yields a factorization in the sense of

Definition 2.1. On the other hand (i) of (H3) is *not* satisfied: there is no choice of C^∞ functions α, β such that

$$\partial_0 [\alpha (x_0(x_2^2 - x_3^2) - x_1) + \beta(2bx_0x_2x_3 - x_4)] \geq 0,$$

since it is easily verified that there is no closed, proper cone $\Gamma(\bar{\rho})$, $\bar{\rho} \in \Sigma_2'$, satisfying condition (3.2) of Proposition 3.4.

C. Let $P_2(x, \xi) = -\xi_0^2 + (x_0 x_2^{2l} - x_1)^2 \xi_n^2 + k_1 \xi_n + ik_2 x_2^l \xi_n$, near $(0, e_n)$, where $k_1 > 0$, $k_2 \in \mathbf{R}$, $l \in \mathbf{N}$. This model satisfies all our assumptions (H1)-(H4). Analogously

$$P_2(x, \xi) = -\xi_0^2 + (x_0(x_2^2 + \xi_2^2 \xi_n^{-2}) - x_1)^2 \xi_n^2 + (x_0 x_3^2 - x_4)^2 \xi_n^2 + k_1(x, \xi'),$$

near $(0, e_n)$, where

$$\mathrm{Im}\, k_1(x, \xi') = O(x_2^2 + \xi_2^2 \xi_n^{-2} + x_3^2),$$
$$\mathrm{Re}\, k_1(x, \xi') > 0,$$

satisfies (H1)-(H4).

D. Let $P_2(x, \xi) = -\xi_0^2 + (x_0 x_2^{2l} - x_1)^2 \xi_n^2 + k(x)\xi_n$, near $(0, e_n)$, where $k(x)$ is a smooth function. After a symplectic dilation in the variables (x_1, ξ_1) we reduce P_2 to the following symbol

$$\tilde{P}_2(x, \xi) = -\xi_0^2 + x_0^2 x_2^{4l} \xi_n^2 + \tilde{k}(x)\xi_n.$$

Recalling the celebrated necessary conditions of Ivrii-Petkov ([9], [8]) we see that in order that the local Cauchy problem for $\tilde{P}_2$ (and hence for P_2) be well-posed we must necessarily have $\tilde{k}(x) = x_2^{2l} \tilde{\tilde{k}}(x)$. Hence if k_1 in Example C is zero, we conclude that Assumption (H4) cannot be improved.

E. In order to give an idea of the microlocal energy estimate needed in the present situation we exhibit an elementary estimate for the operator in Example D. We will not enter into the details of the symbolic calculus and proceed only at a formal level.

Denote by $f^2(x) = (x_0 x_2^{2l} - x_1)^2 + \xi_n^{-1}$; this is the formal time function used in the estimate. The operator P_2 satisfies all our assumptions and f verifies the conditions of Hypothesis (H3). Denote by $\| v \|$ the L^2-norm of a C_0^∞ function v in the open set Ω. Therefore we compute

$$(3.3) \quad 2i\mathrm{Im} \langle f^{-N} P_2 u, -f^{-N} D_0 u \rangle$$
$$= 2i\mathrm{Im} \langle f^{-N}(-D_0^2 + f^2 D_n^2 + D_n + ikx_2^{2l}D_n)u, -f^{-N}D_0u \rangle$$
$$= D_0 \left[\| f^{-N}D_0 u \|^2 + \| f^{-N+1}D_n u \|^2 + \| f^{-N}D_n^{1/2} u \|^2 \right]$$

$$- 2iN\mathrm{Re}\left[\|f^{-N-1/2}x_2^l D_0 u\|^2\right.$$

$$+ \frac{N-1}{N}\|f^{-N-1/2}x_2^l D_n u\|^2 + \left.\|f^{-N-1/2}x_2^l D_n^{1/2} u\|^2\right]$$

$$- 2i\mathrm{Im}\langle f^{-N+1/2}ikx_2^l D_n u, f^{-N-1/2}x_2^l D_0 u\rangle$$

$$= D_0 E_1 - 2iN E_2 + i\mathscr{R}.$$

It is easy to see that $|\mathscr{R}|$ can be estimated by $N E_2$ provided N is sufficiently large and that E_1 and E_2 define positive energies. We also note how the positivity of the real eigenvalue of F_{P_2}, $\lambda = x_2^{2l}\xi_n$, plays a crucial role in establishing the estimate.

Theorem 2.1. is a consequence of the following Proposition:

Proposition 3.5. *The following (local) energy estimate holds*:

$$\int_{-\infty}^0 e^{-2\tau x_0}\|Pu\|_{N+s}^2 dx_0$$

$$> C\tau^2\int_{-\infty}^0 e^{-2\tau x_0}\left[\|u\|_{s+\frac{1}{2}}^2 + \|D_0 u\|_s^2\right] dx_0$$

$$- C_l'\tau\int_{-\infty}^0 e^{-2\tau x_0}\left[\|u\|_{-l}^2 + \|D_0 u\|_{-l}^2\right] dx_0,$$

for every $\forall s \in \mathbf{R}, \forall l \in \mathbf{R}$.

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