A Higher-dimensional Analogue of Carlitz-Drinfeld Theory

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The purpose of this paper is to generalize the arguments of Carlitz and Drinfeld to the higher-dimensional case by giving some analogies of special functions like the Carlitz exponential, the zeta function, the gamma functions, and the modular forms. This is a résumé of my master thesis at University of Tokyo, March 1994, and the details will be published elsewhere.

In the paper of Kapranov [6], the method of the completion is given and the higher-dimensional version of the zeta function is defined. So we apply the idea of Kapranov to define some analogues of the special functions other than the zeta function and deduce the properties of these functions.

1. An analogue of Carlitz exponential. Let $A = A_n = \boldsymbol{F}_q[T_1, \dots, T_n]$ be the polynomial ring over finite field in n variables and $k = k_n = \boldsymbol{F}_q(T_1, \dots, T_n)$ be its field of quotients. The ring A is embedded discretely into the complete topological field $K = K_n = \boldsymbol{F}_q(\!(t_1)\!) \dots (\!(t_n)\!)$ with t_n-adic valuation when we set

$$T_1 = \frac{t_{n-1}}{t_n},\ T_2 = \frac{t_{n-2}}{t_n}, \dots, T_{n-1} = \frac{t_1}{t_n},\ T_n = \frac{1}{t_n}.$$

Let $C = C_n = \hat{\bar{K}}$ be the completion of the algebraic closure of the field K and for any $\boldsymbol{F}_q$-lattice Λ over C we define the function $e_\Lambda : C \to C$

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda - O} \left(1 - \frac{z}{\lambda}\right)$$

where we call any discrete submodule 'lattice'.

The function e_Λ satisfies the following properties.

(1) e_Λ is entire.

(2) e_Λ is $\boldsymbol{F}_q$-linear and Λ-periodic.

(3) e_Λ has simple zeroes at the points of Λ, and no further zeroes.

(4) if $\Lambda, \Lambda' = c\Lambda (c \in C^*)$ are similar lattices, then $ce_\Lambda(z) = e_{\Lambda'}(cz)$.

(5) The derivative satisfies $e'_\Lambda(z) = 1$.

We define the power series $\phi_a^\Lambda(z)$ by $e_\Lambda(az) = \phi_a^\Lambda(e_\Lambda(z))$. In the higher-dimentional case, for an A-module Λ of finite rank, we have $^\#(a^{-1}\Lambda/\Lambda) = \infty$ for any $a \in A - \boldsymbol{F}_q$, and $\phi_a^\Lambda(z)$ is not a polynomial like in the one-dimentional case.

In the two-dimensional case, we have the following theorem.

Theorem 1. *Let $A = A_2 = \boldsymbol{F}_q[X, Y]$ and (X, Y^i) be the ideal of A generated by X and Y^i. Then the coefficients of the series*

$$e_A(Xz) = \sum_{i=0}^{\infty} l_i e_A(z)^{q^i}$$

are written as

$$l_O = X,\ l_i = X^{q^i} \sum_{0 \le j_1 < j_2 \dots < j_i} \tau_{j_1} \tau_{j_2}^q \cdots \tau_{j_i}^{q^{i-1}} \ (i > 0),$$
$$\tau_i = - e_{(X, Y^{i+1})}(Y^i)^{1-q}$$

and their valuations are

$$v(l_i) = q^i + (q-1) \sum_{j=0}^{i-1} \sum_{k=1}^{j} q^{j + \frac{k(k-1)}{2}}.$$

2. The analogue of zeta function. The Goss zeta function was generalized to the case of $A = A_n = \boldsymbol{F}_q[T_1, \dots, T_n]$ by Kapranov [6]. We recall the construction.

We start with the definition of the term 'monic'. For $a \in K$, let $a^{(1)}$ be the element of $\boldsymbol{F}_q(\!(t_1)\!) \dots (\!(t_{n-1})\!)$ such that

$$a = a^{(1)} t_n^{v(a)} + a',\ v(a') > v(a).$$

Similarly, $a^{(2)} \in \boldsymbol{F}_q(\!(t_1)\!) \dots (\!(t_{n-2})\!)$ can be derived from $a^{(1)}$ and finally we get an element $a^{(n)} \in \boldsymbol{F}_q$. In this case, we call an element a 'monic' iff $a^{(n)} = 1$. The set of monic elements is closed under multiplication and this definition is compatible with the old one for $A = A_1$.

For any natural integer s the series

$$\zeta_A(s) = \sum_{\text{monic } a \in A} \frac{1}{a^s}$$

is convergent because the point set $\{a^{-s} \mid a \in A - 0\}$ has at most finite points in neighborhood of 0 and non-Archimedean property shows this. In addition to this, $A = A_n$ is also an UFD as in the case of one-dimensional, then the above sum has the Euler product

$$\prod_{\text{monic irred. } \wp \in A} (1 - \wp^{-s})^{-1}.$$

Kapranov [6] proved that $\zeta_A(s) \in A$ holds for any negative integer s.

The exponential $e_A(z)$ has the relation to the special values of ζ_A.

Theorem 2. *Let*

$$\phi_a^A(z) = \sum_{i=0}^{\infty} l_i z^{q^i},\ \log_A(z) = \sum_{i=0}^{\infty} \beta_i z^{q^i},\ \frac{z}{e_A(z)} = \sum_{i=0}^{\infty} \gamma_i z^i.$$

Then,

$$(a - a^{q^k})\zeta_A(q^k - 1) = \sum_{i=0}^{k-1} l_{k-i}^{q^i} \zeta_A(q^i - 1),$$

$$\zeta_A(q^k - 1) = \beta_k,$$

$$\zeta_A((q-1)k) = \gamma_{(q-1)k}$$

for $k = 1, 2, \ldots$.

Particularly, in two-dimensional case $A = A_2 = \boldsymbol{F}_q[X, Y]$ and $a = X$, the coefficients

$$l_0 = X,\ l_i = X^{q^i} \sum_{0 \leq j_1 < j_2 < \ldots < j_i} \tau_{j_1} \tau_{j_2}^q \cdots \tau_{j_i}^{q^{i-1}}\ (i > 0),$$

$$\tau_i = -e_{(X, Y^{i+1})}(Y^i)^{1-q}$$

have been derived, and they give the special values of zeta function and especially

$$\zeta_{A_2}(q-1) = \frac{1}{1 - X^{1-q}} \sum_{i=0}^{\infty} e_{(X, Y^{i+1})}(Y^i)^{1-q}$$

holds.

3. Some analogues of gamma functions. Now, we will generalize the gamma functions to the case of $A = A_n = \boldsymbol{F}_q[T_1, \ldots, T_n]$.

Let the definition of 'monic' be the same as in the argument of the zeta function and let

$$D_i = \prod_{\substack{\text{monic } a \in A \\ \deg a = i}} a.$$

Then, we express any nonnegative integer n as $\sum n_i a^i$, $0 \leq n_i \leq q - 1$ and set

$$\Pi(n) = \prod D_i^{n_i} \in A$$

and define the first gamma function as $\Gamma(n) = \Pi(n-1)$. This function satisfies the following property.

Theorem 3. *Let a and b be any nonnegative integers, then*

$$\frac{\Pi(a+b)}{\Pi(a)\Pi(b)} \in A.$$

Now we define $\langle a \rangle$ for a monic $a \in K$, we set

$$\langle a \rangle = a t_n^{-v_{t_n}(a)} t_{n-1}^{-v_{t_{n-1}}(a^{(1)})} \ldots t_1^{-v_{t_1}(a^{(n-1)})}$$

where v_{t_i} is t_i-adic valuation of $\boldsymbol{F}_q((t_1)) \ldots ((t_i))$. In this case, $\langle a \rangle$ is integral in the mean of t_i-adic valuation and $\langle a \rangle^{(i)}$ is also integral in the mean of t_{n-i} valuation and finally $\langle a \rangle^{(n)} = 1$ satisfies. We call such an element of K absolutely integral. In this case, for any $b \in K$, to be absolutely integral is equivalent to be monic and satisfy the condition $\langle b \rangle = b$. As the set of absolutely integral elements is closed under multiplication, $\langle ab \rangle = \langle a \rangle \langle b \rangle$ satisfies.

And now, for any $z = \sum_{i=0}^{\infty} z_i q^i \in \boldsymbol{Z}_p$, we consider the following gamma function

$$\Pi_\infty(z) = \prod_{i=0}^{\infty} \langle D_i \rangle^{z_i},$$

$$\Gamma_\infty(z) = \Pi_\infty(z - 1).$$

Theorem 4. *$\Gamma_\infty(z)$ satisfies the following properties.*

(1) $\Gamma_\infty(z)\Gamma_\infty(1-z) = \Gamma_\infty(0)$.

(2) *If $p \nmid n$, then* $\Gamma_\infty(z)\Gamma_\infty\left(z + \frac{1}{n}\right)\ldots$

$$\Gamma_\infty\left(z + \frac{n-1}{n}\right)/\Gamma_\infty(nz) = \Gamma_\infty(0)^{\frac{n-1}{2}}.$$

Next, we define one more gamma function of characteristic p. We put $A_{\leq 0} = \{-a \in A \mid a$ is monic or $0\}$ and for any $z \in C - A_{\leq 0}$, we define the gamma function $\Gamma_0(z)$ as

$$\Gamma_0(z) = \frac{1}{z} \prod_{\text{monic } a \in A} \left(1 + \frac{z}{a}\right)^{-1}$$

and define the factorial $\Pi_0(z)$ as $\Pi_0(z) = z\Gamma_0(z)$. This function satisfies the following property.

Theorem 5. *The factorial function $\Pi_0(z)$ satisfies*

$$\prod_{c \in A^* = \boldsymbol{F}_q^*} \Pi_0(cz) = \frac{z}{e_A(z)}.$$

Remarks. In general, the fact that $1/\Pi_0(a) \in A$ for any $a \in A$ does not hold in the higher-dimensional case. In fact,

$$\frac{1}{\Pi_0(Y)} = \frac{2(Y+1)(X^q - X + Y^q - Y)}{X^q - X}$$

occurs in the case of $A = A_2 = \boldsymbol{F}_q[X, Y]$.

4. Modular forms. Now, we take the space which corresponds to the classical upper and lower half-plane as

$$\Omega = C - K.$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$ act on $z \in \Omega$ via

$$\gamma z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az + b}{cz + d}.$$

We define a modular form of weight k as a function $f : \Omega \to C$ which satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in = GL_2(A)$. Let M_k be the C-vector space of modular forms of weight k, then $M_k M_{k'} \subset M_{k+k'}$ holds and $M_k \neq 0$ only if $(q-1) \mid k$.

Now, we define the Eisenstein series as an example of modular forms.

Theorem 6. *The Eisenstein series*

$$E^{(k)}(z) = \sum_{(m,n)\in A^2-(0,0)} (mz+n)^{-k}$$

is a modular form of weight k.

As in the one-dimensional case, modular forms are related to rank 2 lattices.

Theorem 7. *Let* $z \in \Omega$ *and take an* A*-lattice of rank 2 with period* $(z, 1)$ *as* $Y_z = Az \oplus A$ *and let*

$$e_{Y_z}(x) = x \prod_{\lambda \in Y_z - 0} \left(1 - \frac{x}{\lambda}\right)$$

which is Y_z*-periodic. Let the power series*

$$\phi_a^{Y_z}(x) = \sum_{k=0}^{\infty} l_k x^{q^k}$$

satisfying $e_{Y_z}(ax) = \phi_a^{Y_z}(e_{Y_z}(x))$ *for any* $a \in A$. *Then* $l_k = l_k(z)$ *is a modular form of weight* $q^k - 1$.

For an $\boldsymbol{F}_q$-lattice $\Lambda \subset C$, let $S_k = \sum_{\lambda\in\Lambda}(z+\lambda)^{-k}$ and $t(z) = 1/e_\Lambda(z)$. In general, $S_k = G_k(t)$ is a polynomial of $t = S_1$. (This corresponds to the Goss polynomial in the one-dimensional case.)

For any $a \in A$ let

$$t_a = t(az) = \frac{1}{e_A(az)}$$

then in the case of $(q-1) \mid k$, the Eisenstein series $E^{(k)}(z)$ can be written as

$$\begin{aligned} E^{(k)}(z) &= \sum_{(a,b)\in A^2-(0,0)} (az+b)^{-k} \\ &= \sum_{b\in A-0} b^{-k} - \sum_{\text{monic } a} \sum_{b\in A} (az+b)^{-k} \\ &= -\zeta_A(k) - \sum_{\text{monic } a} G_k(t_a). \end{aligned}$$

However, different from the one-dimensional case, in general, $t_a = t(az)$ can not be written as the power series of $t = t(z) = 1/e_A(z)$. In fact, let $\{z_i\}$ be a sequence such that $e_A(z_i) \to 0$, then putting $z_i' = T_2^i/T_1 + z_i$, we have

$$e_A(z_i') = e_A(T_2^i/T_1) + e_A(z_i),$$
$$e_A(T_1 z_i') = e_A(T_2^i) + e_A(T_1 z_i) = \phi_{T_1}^A(e_A(z_i)).$$

Therefore, $t_{T_1} \to \infty$ occurs even if $t \to 0$.

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