# On Some Exceptional Rational Maps

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**§1. Introduction.** Throughout this paper let $\hat{C} = C \cup \{\infty\}$ be the Riemann sphere and $R : \hat{C} \to \hat{C}$ be a rational map of degree $d \geq 2$. Periodic orbits are one of the most important objects to study in the theory of dynamical systems. By definition a point $z_0$ is an *n-periodic point* if

$$R^n(z_0) := \overbrace{R \circ \cdots \circ R}^{n}(z_0) = z_0$$
$$\text{and } R^k(z_0) \neq z_0 \text{ for } k < n$$

and we call the set $\{z_0, R(z_0), R^2(z_0), \ldots, R^{n-1}(z_0)\}$ an *n-periodic orbit.* It is a fundamental problem to ask if there exists at least one $n$-periodic orbit of $R$ for each natural number $n$, or equivalently, in which case rational maps fail to have $n$-periodic orbits for some $n$. For this problem the following result is well known.

**Theorem** ([1]). *Let $R$ be a rational map of degree $d \geq 2$. If $R$ has no $n$-periodic orbit, then the pair $(d, n)$ is either* (2,2), (2,3), (3,2) *or* (4,2).

From this theorem one can say that "most"rational maps have at least one $n$-periodic orbit for any $n$ and a rational map which does not satisfy this property is rather "exceptional". In [1] Baker also constructed concrete examples for each pair of $(d, n)$ above to show that the exceptional maps really exist. Now, how many kinds of such maps exist ? In this paper we answer this question, that is, we investigate whole these exceptional rational maps and completely classify them up to the conjugation by Möbius transformations. Also we give an inequality that holds among the numbers of $n$-periodic orbits and give a lower estimate of the number of $p$-periodic orbits in the case where $p$ is prime.

For the definitions of the concepts we do not mention here, see for example [2].

**§2. Normal forms and a key lemma.** In order to classify all the exceptional rational maps, it is sufficient to determine all the conjugacy classes and give each representative element. For this purpose we construct normal forms for each $d(= 2,3,$ or $4)$ and find the representatives in it. If $R$ is an exceptional rational map, all the 2-periodic orbits (or 3-periodic orbits in the case of $(d, n) = (2,3)$) are considered to be degenerate to some fixed points (or equivalently 1-periodic orbits). Therefore it is convenient to construct normal forms which specify some fixed points. Since one can easily see that each exceptional rational map has at least two distinct fixed points, we may assume that 0 and $\infty$ are such points by the conjugacy of the Möbius transformation which maps these two distinct fixed points to 0 and $\infty$. With one more suitable Möbius transformation for each cases, we obtain the following normal forms. Here, for example, $m(0\,; R)$ denotes the multiplier of $R$ at the fixed point $z = 0$.

**Proposition.** *Every rational map of degree 2, 3,* or *4 which admits at least two distinct fixed points is conjugate to the following normal forms, respectively.*

(i) $d = 2 \quad R(z) = \dfrac{z^2 + bz}{az + 1}, \quad a = m(\infty\,; R),$
$b = m(0\,; R).$

(ii) $d = 3 \quad R(z) = \dfrac{z^3 + cz^2 + ez}{az^2 + bz + 1},$
$a = m(\infty\,; R), \quad e = m(0\,; R).$

(iii) $d = 4 \quad R(z) = \dfrac{z^4 + ez^3 + fz^2 + gz}{az^3 + bz^2 + cz + 1},$
$a = m(\infty\,; R), \quad g = m(0\,; R).$

The next lemma is a key for solving our problem.

**Lemma.** *Suppose that $R(z_0) = z_0$. Let $mul(z_0\,; R)$ denotes the multiplicity of the fixed point $z = z_0$. For a fixed natural number $n$,*

(i) *if $\{m(z_0\,; R)\}^n \neq 1$, then $mul(z_0\,; R) = mul(z_0\,; R^n)$.*

*In the case of $\{m(z_0\,; R)\}^n = 1$,*

(ii) *if $m(z_0\,; R) = 1$, then $mul(z_0\,; R) = mul(z_0\,; R^n) > 1$.*

(iii) *If $m(z_0\,; R)$ is a primitive $t$-th root of unity, then $t \mid n$ and $mul(z_0\,; R) = 1$ and $mul(z_0\,;$*

$R^n) = kt + 1$ *for some* $k \in \boldsymbol{N}$.

**§3. Main results.** Using the normal forms in the Proposition and the Lemma, we can prove the following theorem.

**Theorem 1.** *Any exceptional rational map is conjugate to one of the following rational maps by a suitable Möbius transformation.*

(1) $(d, n) = (2,2)$

$$R_1(z\,;a) = \frac{z^2 - z}{az + 1} \quad (a \neq -1),$$

*moduli space* $= (\boldsymbol{C}\backslash\{-1\})/_{\sim} \cong \boldsymbol{C}$.

(2) $(d, n) = (2,3)$

(i) *(1 + 1 + 7)-type*

$$R_2(z) = \frac{z^2 + \omega z}{\frac{\omega + 5}{\omega - 1}z + 1}, \quad R_3(z) = \frac{z^2 + \omega^2 z}{\frac{\omega^2 + 5}{\omega^2 - 1}z + 1},$$

*where* $\omega = \dfrac{-1 + \sqrt{-3}}{2}$.

(ii) *(1 + 4 + 4)-type*

$$R_4(z) = \frac{z^2 + \omega z}{\omega z + 1}, \quad R_5(z) = \frac{z^2 + \omega^2 z}{\omega^2 z + 1}.$$

(3) $(d, n) = (3,2)$

(i) *(1 + 1 + 1 + 7)-type*

$$R_6(z\,;c) = \frac{z^3 + cz^2 - z}{(c^2 - 1)z^2 - 2cz + 1} \quad (c \neq 0),$$

*moduli space* $= (\boldsymbol{C}\backslash\{0\})/_{\sim} \cong \boldsymbol{C}\backslash\{0\}$.

(ii) *(1 + 1 + 3 + 5)-type*

$$R_7(z\,;b) = \frac{z^3 - z}{-z^2 + bz + 1} \quad (b \neq 0),$$

*moduli space* $= (\boldsymbol{C}\backslash\{0\})/_{\sim} \cong \boldsymbol{C}\backslash\{0\}$.

(iii) *(1 + 3 + 3 + 3)-type*

$$R_8(z\,;b) = \frac{z^3 + \frac{4}{b}z^2 - z}{-z^2 + bz + 1} \quad (b \neq 0, \pm 2i),$$

*moduli space* $= (\boldsymbol{C}\backslash\{0, \pm 2i\})/_{\sim} \cong \boldsymbol{C}$.

(4) $(d, n) = (4,2)$

(i) *(1 + 1 + 1 + 1 + 13)-type*

$$R_9(z) = \frac{z^4 - z}{-2z^3 + 1}.$$

(ii) *(1 + 1 + 1 + 5 + 9)-type*

$$R_{10}(z) = \frac{z^4 + z^3 + z^2 - z}{-z^3 + z^2 - 3z + 1}.$$

(iii) *(1 + 1 + 5 + 5 + 5)-type*

$$R_{11}(z) = \frac{z^4 + \bar{c}_0 z^3 + \bar{b}_0 z^2 - z}{-z^3 + b_0 z^2 + c_0 z + 1},$$

*where* $b_0 = \dfrac{3 + \sqrt{5}}{2}$, $\bar{b}_0 = \dfrac{3 - \sqrt{5}}{2}$,

$$c_0 = \frac{-5 - \sqrt{5}}{2}, \ \bar{c}_0 = \frac{-5 + \sqrt{5}}{2}.$$

(iv) *(1 + 3 + 3 + 3 + 7)-type*

$$R_{12}(z) = \frac{z^4 - 3^{\frac{1}{3}}z^3 + 3^{\frac{2}{3}}z^2 - z}{-z^3 + 3^{\frac{4}{3}}z^2 - 5\cdot 3^{-\frac{1}{3}}z + 1}.$$

Here, by a *moduli space* of a parameter space we mean a space obtained by taking the quotient of the parameter space with the following equivalence relation:

$a \sim a' \Leftrightarrow R(z\,;a)$ is conjugate to $R(z\,;a')$ by a Möbius transformation.

As for the definition of the *type*, we will explain below.

*Outline of the proof.* Here we only sketch the outline of the proof of (1) and (2). The case (3) and (4) can be proved by the same method with much more tedious and complicated calculation.

(1) Since, generally, $R$ has $d + 1$ fixed points with counting multiplicity, $R$ has three and $R^2$ has five fixed points in this case. Among these five fixed points of $R^2$, generically, two of them are not fixed points of $R$ and the set consisting of these two points is a unique 2-periodic orbit of $R$. If $R$ has no 2-periodic orbit, then the unique 2-periodic orbit is degenerate to a fixed point. We may assume that $z = 0$ is such a point without loss of generality. Then by the Lemma we have $b = m(0\,;R) = -1$ in the normal form and since $d = 2$ we have $a \neq -1$. Thus $R_1(z\,;a) = \dfrac{z^2 - z}{az + 1}$ $(a \neq -1)$ are the desired rational maps.

Next we compute the moduli space. All the fixed points of $R(z\,;a)$ are $0$, $\infty$, and $\dfrac{2}{1 - a}$ (when $a = 1$ this point is equal to $\infty$ and $mul(\infty\,;R) = 2$) and their multipliers are $-1$, $a$, and $\dfrac{3 - a}{1 + a}$ respectively . If $R(z\,;a) \sim R(z\,;a')$, then since the multipliers are invariant by a conjugation with Möbius transformations, we have

$$a' = a \text{ or } a' = \frac{3 - a}{1 + a} \ (=: g(a)).$$

It is easy to see that $R(z\,;a) \sim R(z\,;a')$ if and only if $a$ and $a'$ are related as above. Since $g \circ g \equiv id$, taking the quotient by the action of the group $G := \{id, g\}$, the moduli space is

$$(\boldsymbol{C}\backslash\{-1\})/G \cong \boldsymbol{C}.$$

(2) A rational map of degree 2 generically

has three fixed points and two 3-periodic orbits. But since $R$ has no 3-periodic orbit, each of two orbits is degenerate to a certain fixed point. There are two cases to consider, one is that these two orbits are degenerate to a same fixed point, say $z = 0$, and the other is that these orbits are degenerate to distinct fixed points, say $z = 0$ and $z = \infty$. Since the multiplicity of the degenerate fixed point of $R^3$ is $3k + 1$ for some $k \in \boldsymbol{N}$ by the Lemma and $R^3$ has nine fixed points, $mul(0\,; R) = 1$ and $mul(0\,; R^3) = 7$ for the former, and $mul(0\,; R) = mul(\infty\,; R) = 1$ and $mul(0\,; R^3) = mul(\infty\,; R^3) = 4$ for the latter. We call such $R$ $(1 + 1 + 7)$-type or $(1 + 4 + 4)$-type, respectively. It is obvious that two different types of rational maps are never conjugate each other.

(i) $(1 + 1 + 7)$-type: Suppose that $mul(0\,; R^3) = 7$, then the by Lemma we have $b = \omega := \dfrac{-1 + \sqrt{-3}}{2}$ or $\omega^2$. Assume that $b = \omega$, then $R^3(z) = z + \beta_4 z^4 + \cdots$ in a small neighborhood of $z = 0$ and again by the Lemma it is necessary and sufficient that $\beta_4 = 0$. The value $\beta_4$ is described by the value $a = m(\infty\,; R)$ and we have $a = \omega^2$ or $\dfrac{\omega + 5}{\omega - 1}$. Since $d = 2$ we have $a \neq \omega^2$ and the result follows. These two obtained rational maps are not conjugate each other, because the sets of multipliers at the three fixed points do not coincide.

(ii) $(1 + 4 + 4)$-type: Suppose that $mul(0\,; R^3) = mul(\infty\,; R^3) = 4$, then by the Lemma we have

$(m(0\,; R), m(\infty\,; R))$
$= (\omega, \omega), (\omega^2, \omega^2), (\omega^2, \omega)$, or $(\omega, \omega^2)$.

Since $d = 2$, only $(\omega, \omega)$ and $(\omega^2, \omega^2)$ are appropriate. These two rational maps are not conjugate for the same reason as in the case of $(1 + 1 + 7)$-type. $\square$

**Remarks.** (1) The essential point of Theorem 1 lies in making clear how many kinds of exceptional rational maps exist. The concrete forms of the maps presented are not so important. Apparently simpler forms could be obtained, if desired. For example, $\dfrac{z^4 + z}{2z^3 - 1}$ is an exceptional rational map of $(1 + 3 + 3 + 3 + 7)$-type.

(2) In the case of $(d, n) = (2,3)$, let $\sigma(z) := \bar{z}$, we have $R_3 = \sigma \circ R_2 \circ \sigma^{-1}$ (resp. $R_5 = \sigma \circ R_4 \circ \sigma^{-1}$). Therefore the dynamics of $R_2$ and $R_3$ (reap. $R_4$ and $R_5$) are essentially the same, though both are not holomorphically conjugate.

(3) In the case of $(d, n) = (4,2)$, at a glance, it seems that there is a possibility of existence of other types of $R$, namely for example, $(1 + 1 + 1 + 3 + 11)$-type, $(1 + 1 + 3 + 3 + 9)$-type, etc. But surprisingly these types of $R$ never exist, although there exist all possible types of $R$ for the other three cases.

(4) As we mentioned in §1, Baker [1] constructed examples of exceptional rational maps for each pair of $(d, n)$. The correspondence between them and our normal forms is the following. In Table I, if the conjugacy is $\varphi(z)$ and the normal form is $R(z)$ then $\varphi \circ R \circ \varphi^{-1}(z)$ is equal to the corresponding Baker's example. (Note that there is a miss print in the Baker's example for the case $(d, n) = (4,2)$. His example was $\dfrac{-z(1 + 2z^2)}{1 - 3z^3}$ but it seems that this is $\dfrac{-z(1 + z^3)}{1 - 2z^3}$.)

Table I

| $(d, n)$ | Baker's example | Type | Normal Form | Conjugacy |
|---|---|---|---|---|
| (2,2) | $z^2 - z$ | $(1 + 1 + 3)$ | $R_1(z\,; 0)$ | id |
| (2,3) | $z + (\omega - 1)\dfrac{z^2 - 1}{2z}$ | $(1 + 4 + 4)$ | $R_4(z)$ | $-\dfrac{z + 1}{z - 1}$ |
| (3,2) | $\dfrac{2}{3z^2} + \dfrac{z}{3}$ | $(1 + 3 + 3 + 3)$ | $R_8(z\,; -2\omega^2)$ | $\dfrac{z + 1}{z + \omega^2}$ |
| (4,2) | $\dfrac{-z(1 + z^3)}{1 - 2z^3}$ (?) | $(1 + 3 + 3 + 3 + 7)$ | $R_{12}(z)$ | $\dfrac{2^{\frac{1}{3}}z}{z - 3^{\frac{1}{3}}}$ |

Let $N_d(m)$ be the number of $m$-periodic orbits of the rational map $R$ of degree $d \geq 2$ with counting multiplicity. If $R$ is exceptional, for example $R(z\,;a) := \dfrac{z^2 - z}{az + 1}$ $(a \neq -1)$ as in the case of $(d, n) = (2,2)$, we have $N_2(2) = 0$ although $N_2(2) = 1$ generically. If $R$ is not exceptional, however, it always holds that $N_d(m) \geq 1$ by the Baker's theorem. That is, no matter how much $R$ is degenerate compared with the generic case, $N_d(m)$ never be 0. Then how can we estimate $N_d(m)$ from below? For this we have the following.

**Theorem II.**

$$d^n - d - \sum_{\substack{i<m\leq n \\ m|n}} mN_d(m) \leq 2(d-1)n.$$

*In particular when $n = p$ with $p$ being prime, we have,*

$$\frac{d^p - d}{p} - 2(d-1) \leq N_d(p). \qquad \square$$

**Remark.** Generically $N_d(p) = \dfrac{d^p - d}{p}$ and $2(d-1)$ is a number of critical points of $R$ with counting multiplicity.

## References

[1] I. N. Baker: Fixpoints of polynomials and rational functions. J. London Math. Soc., **39**, 615–622 (1964).

[2] J. Milnor: Dynamics in one complex variable. Introductory Lectures. SUNY Stony Brook (1990/5) (preprint).