Note on Siegel-Eisenstein Series

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1. Siegel-Eisenstein series. In this paper, we will treat two types of Eisenstein series and give some remarks. Let $\boldsymbol{H}_n$ be the Hermitian upper half space of degree n, namely, the domain consisting of all complex square matrices of size n such that the Hermitian imaginary part $\Im(Z) := (2i)^{-1}(Z - \bar{Z}^T)$ is positive definite. Here $\bar{Z}^T$ is the transpose, complex conjugate matrix of Z. The Siegel upper half space $\boldsymbol{S}_n := \{Z \in \boldsymbol{H}_n \mid Z^T = Z\}$ is a submanifold of $\boldsymbol{H}_n$. If $Z \in \boldsymbol{S}_n$, then $I(Z) := \Im(Z)$ is exactly equal to the imaginary part of Z. Consider an imaginary quadratic field K of discriminant d_K. The ring of integers in K is denoted by $\mathcal{O} = \mathcal{O}_K$. The Hermitian modular group of degree n associated with K is defined as:

$$\Gamma_n(K) := \left\{M \in SL_{2n}(\mathcal{O}) \mid \bar{M}^T J_n M = J_n, J_n = \begin{pmatrix} 0_n & E_n \\ -E_n & 0_n \end{pmatrix}\right\}.$$

The Siegel modular group of degree n is defined as $\Gamma_n := Sp_n(\boldsymbol{Z})$. Let $[\Gamma_n, k]$ (resp. $[\Gamma_n(K), k]$) be the vector space of holomorphic Siegel modular forms (resp. Hermitian modular forms) of weight k for Γ_n (resp. $\Gamma_n(K)$).

Let us consider the Eisenstein series of the following two types:

(*SP Case*)

$$E_k^{(n)}(Z, s) := \det I(Z)^s \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_{n,0} \backslash \Gamma_n} \det(CZ + D)^{-k} \mid \det(CZ + D) \mid^{-2s}, \ Z \in \boldsymbol{S}_n$$

(*SU Case*)

$$E_{k,K}^{(n)}(Z, s) := \det \Im(Z)^s \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_n(K)_0 \backslash \Gamma_n(K)} \det(CZ + D)^{-k} \mid \det(CZ + D) \mid^{-2s}, \ Z \in \boldsymbol{H}_n$$

Here k is an even integer and $\Gamma_{n,0}$ (resp. $\Gamma_n(K)_0$) is the subgroup of Γ_n (resp. $\Gamma_n(K)$) consisting of the elements $M = \begin{pmatrix} A & B \\ 0_n & D \end{pmatrix}$ in Γ_n (resp. $\Gamma_n(K)$). It is known that $E_k^{(n)}(Z, s)$ (resp. $E_{k,K}^{(n)}(Z, s)$) is convergent for $\mathrm{Re}(s) > (n + 1 - k)/2$ (resp. $\mathrm{Re}(s) > (2n - k)/2$). Moreover, they can be continued as meromorphic functions in s to the whole complex plane. The analytic properties of these Eisenstein series were successfully studied by Shimura [5] and Weissauer [6]. In fact, Shimura found the following results.

Theorem 1 (Shimura). (1) (*SP Case*) $E_{\frac{n-1}{2}}^{(n)}(Z, s)$ *has at most a simple pole at* $s = 1$. *The residue at* $s = 1$ *is* π^{-n} *times an element* f *in* $\left[\Gamma_n, \frac{n-1}{2}\right]$ *with rational Fourier coefficients.*

(2) (*SU Case*) $E_{n-1,K}^{(n)}(Z, s)$ *has at most a simple pole at* $s = 1$. *The residue at* $s = 1$ *is* π^{-n} *times an element* f *in* $[\Gamma_n(K), n - 1]$ *with rational Fourier coefficients.*

Remark 1. The definition of Eisenstein series in [5] is slightly different from our definition. The Eisenstein series Shimura treated were $\det I(Z)^{-\frac{s}{2}} E_k^{(n)}\left(Z, \frac{s}{2}\right)$ (SP Case) and $\det \Im(Z)^{-\frac{s}{2}} E_{k,K}^{(n)}\left(Z, \frac{s}{2}\right)$ (SU Case) in our notation.

2. A residue formula. Our purpose is to specify the modular forms f in Theorem 1. The first result is as follows:

Theorem 2. (1) *For any even, positive integer* k *such that* $k < \frac{n+1}{2}$, $E_k^{(n)}(Z, s)$ *is holomorphic in* s *at* $s = 0$ *and* $E_k^{(n)}(Z, 0)$ *defines an element of* $[\Gamma_n, k]$ *with rational Fourier coefficients.*

(2) *Assume that the class number of* K *is* 1. *For any even, positive integer* k *such that* $k < n$, $E_{k,K}^{(n)}(Z, s)$ *is holomorphic in* s *at* $s = 0$ *and* $E_{k,K}^{(n)}(Z, 0)$ *defines an element of* $[\Gamma_n(K), k]$ *with rational Fourier coefficients.*

A proof of (1) was already given in Weissauer [6]. Another proof is found by using results of Arakawa [1] and Mizumoto [3].

Here we must introduce the following notation:

$$\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

$$\xi(s; \chi_K) := \pi^{-\frac{s}{2}} \mid d_K \mid^{\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s; \chi_K),$$

$$\xi_K(s) := \xi(s)\xi(s\,;\chi_K) = \pi^{-s}\,|\,d_K\,|^{\frac{s}{2}}\,\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big)\zeta_K(s),$$

where $\Gamma(s)$: the gamma function, $\zeta(s)$: the Riemann zeta function, $L(s\,;\chi_K)$: the Dirichlet L-function associated with the Kronecker character χ_K, $\zeta_K(s)$: the Dedekind zeta function of K.

Theorem 3. (1) (*SP Case*) *Let m and n be integers satisfying $0 \le m < n$, $n \equiv m$* (mod 4). *Then the residue of $E^{(n)}_{\frac{n-m}{2}}(Z, s)$ at $s = \frac{m+1}{2}$ is given by*

$$\operatorname*{Res}_{s=\frac{m+1}{2}} E^{(n)}_{\frac{n-m}{2}}(Z, s) = (-1)^{\frac{n(n-m)}{4}} 2^{-2} \times \prod_{j=0}^{\frac{n-m-4}{4}} \frac{(m+1+2j)!\Big(\frac{n-m}{2}+2j\Big)!}{(2j)!\Big(\frac{n+m}{2}+1+2j\Big)!} \times \frac{\xi\Big(\frac{n-m}{2}\Big)}{\xi\Big(\frac{n+m}{2}+1\Big)} \frac{\prod_{i=0}^{m-1}\xi(i+2)}{\prod_{l=0}^{m}\xi(n+m-2l)} E^{(n)}_{\frac{n-m}{2}}(Z, 0).$$

(2) (*SU Case*) *Assume that the class number of K is* 1. *Let m and n be integers satisfying $1 \le m < n$, $n \equiv m$* (mod 2). *Then the residue of $E^{(n)}_{n-m,K}(Z, s)$ at $s = m$ is given by*

$$\operatorname*{Res}_{s=m} E^{(n)}_{n-m,K}(Z, s) = (-1)^{\frac{(n+1)(n-m)}{2}} 2^{-1} \times \prod_{j=0}^{\frac{n-m-2}{2}} \frac{(m+j)!\Big(\frac{n-m}{2}+j\Big)!}{j!\Big(\frac{n+m}{2}+j\Big)!} \times \frac{\xi(1\,;\chi_K)\prod_{i=0}^{m-2}\xi_K(i+2)}{\prod_{l=0}^{2m-1}\xi(n+m-l\,;\chi_K^l)} E^{(n)}_{n-m,K}(Z, 0).$$

Here we understand that $\xi(s\,;\chi_K^m) = \xi(s)$ if m is even; $= \xi(s\,;\chi_K)$ *if m is odd.*

Using the theory of singular modular forms, we can get the following corollaries:

Corollary 1. (1) (*SP Case*) *If $0 \le m < n$ and $n \equiv m + 4$* (mod 8), *then $E^{(n)}_{\frac{n-m}{2}}(Z, s)$ is holomorphic at $s = \frac{m+1}{2}$.*

(2) (*SU Case*) *Assume that the class number of K is* 1. *If $1 \le m < n$ and $n \equiv m + 2$* (mod 4), *then $E^{(n)}_{n-m,K}(Z, s)$ is holomorphic at $s = m$.*

Corollary 2. (1) (*SP Case*)

$$\operatorname*{Res}_{s=1} E^{(n)}_{\frac{n-1}{2}}(Z, s) = \pi^{-n}\Big\{(-1)^{\frac{n-1}{4}} 2^{-2n-3}(n+1)!(n+1)(n+3) \times \frac{B_2 B_{\frac{n-1}{2}}}{B_{\frac{n+3}{2}} B_{n+1} B_{n-1}} E^{(n)}_{\frac{n-1}{2}}(Z, 0)\Big\}.$$

(2) (*SU Case*)

$$\operatorname*{Res}_{s=1} E^{(n)}_{n-1,K}(Z, s) = \pi^{-n} \times \Big\{-\frac{2^{1-2n}\,|\,d_K\,|^{\frac{n-1}{2}}}{w_K}\frac{(n+1)!\,n}{B_{n+1}B_{n,\chi_K}} E^{(n)}_{n-1,K}(Z, 0)\Big\}.$$

Here B_n and $B_{n,\chi}$ are the n-th Bernoulli number and the generalized Bernoulli number respectively, and w_K the order of the unit group of K.

Remark 2. We take the definition of B_n, $B_{n,\chi}$ from [2], p. 89, p.94 respectively.

Remark 3. In the special case $\boldsymbol{K} = \boldsymbol{Q}(i)$, $n = 5$, the residue formula in (2) of Corollary 2 was already given in [4]. The Eisenstein series treated there is $\tilde{E}^{(n)}_{k,K}(Z, s) := \det \Im(Z)^{-\frac{s}{2}} E^{(n)}_{k,K}\Big(Z, \frac{s}{2}\Big)$ in our notation. The residue formura in [4] was

$$\operatorname*{Res}_{s=2} \tilde{E}^{(5)}_{4,K}(Z, s) = \frac{\pi^6\,|\,d_K\,|^{-\frac{5}{2}}\det\Im(Z)^{-1}}{\Gamma(5)\zeta(6)L(5\,;\chi_K)}\theta^{(5)}(Z\,;I)$$

where $\theta^{(5)}(Z\,;I) = \sum_X \exp[\pi i \mathrm{tr}(\bar{X}^T IXZ)]$ is the theta series associated with Iyanaga's matrix I (for the precise definition, see [4], p. 117). Since $\theta^{(5)}(Z\,;I) = 2^{-1}\tilde{E}^{(5)}_{4,K}(Z, 0) = 2^{-1}E^{(5)}_{4,K}(Z, 0)$, we have

$$\operatorname*{Res}_{s=1} E^{(5)}_{4,K}(Z, s) = 2^{-1}\det\Im(Z)\operatorname*{Res}_{s=2}\tilde{E}^{(5)}_{4,K}(Z, s) = 2^{-1}\frac{(2!)^2}{4!}\frac{\xi(1\,;\chi_K)}{\xi(6)\xi(5\,;\chi_K)}E^{(5)}_{4,K}(Z, 0)$$

This shows (2) in Corollary 2 in the special case. Finally, we note that there is a minor mistake in [4]. In the final formula (3) in [4](p. 117), the factor $|\,d_K\,|^{\frac{5}{2}}$ should be $|\,d_K\,|^{-\frac{5}{2}}$.

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