Positive Solution of Some Nonlinear Elliptic Equation with Neumann Boundary Conditions*)

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Abstract: In this note we show that there exists Λ_0 such that, for every $\lambda \in (0, \Lambda_0)$, the problem: $-\Delta u = \lambda u^q + W(x)u^p$ in Ω, $u > 0$ in Ω, $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, where $\Omega \subset R^N$ is a bounded convex domain with smooth boundary, $0 < q < 1 < p$ and $W \in C^1(\bar{\Omega})$, has a solution u_λ iff $\int_\Omega W(x)dx < 0$. Moreover: $\| u_\lambda \|_\infty \to 0$ as $\lambda \downarrow 0$.

1. Introduction. In this note we study the Neumann problem for a class of semilinear elliptic equations.

Let $\Omega \subset R^N$ be a bounded convex domain with smooth boundary $\partial\Omega$ and consider the semilinear elliptic problem:

$$(1_\lambda) \begin{cases} -\Delta u = \lambda u^q + W(x)u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < q < 1 < p$ and $W \in C^1(\bar{\Omega})$. The influence of negative part of W is displayed in the following condition:

$$(*) \qquad \int_\Omega W(x)dx < 0.$$

As it turns out, condition $(*)$ was inspired by a corresponding necessary condition derived in [2]. The corresponding Dirichlet problem:

$$\begin{cases} -\Delta u = \lambda u^q + u^p & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases}$$

with $0 < q < 1 < p$, has been extensively studied in the paper of Ambrosetti, Brezis and Cerami [1]. Moreover, by the results of Boccardo, Escobedo and Peral [4], these results are extended for the p-laplacian. The purpose of the present note is to study (1_λ) and our main result is the following:

Theorem 1.1. If $(*)$ is satisfied, then there exists $\Lambda_0 \in R$, $\Lambda_0 > 0$, such that, for all $\lambda \in (0, \Lambda_0)$, problem (1_λ) has a solution u_λ and $\| u_\lambda \|_\infty \to 0$ as $\lambda \downarrow 0$.

The proof of the above theorem uses only elementary tools. It is based on the construction of explicit sub and super solutions for (1_λ) and the application of the Sattinger results (see [6]).

2. The existence result.

Lemma 2.1. Suppose there exists $\lambda > 0$ such that the problem (1_λ) has a solution u_λ. Then necessarily the condition $(*)$ must hold.

Proof. For each $\varepsilon > 0$ put:

$$f_\varepsilon(u_\lambda) = \frac{1}{1-p}(u_\lambda + \varepsilon)^{1-p}.$$

We observe that:

$$-\Delta f_\varepsilon(u_\lambda) = (u_\lambda + \varepsilon)^{-p}(\lambda u_\lambda^q + W(x)u_\lambda^p) + p(u_\lambda + \varepsilon)^{-p-1}|\nabla u_\lambda|^2 \text{ in } \Omega,$$

$$\frac{\partial f_\varepsilon(u_\lambda)}{\partial n} = (u_\lambda + \varepsilon)^{-p}\frac{\partial u_\lambda}{\partial n} = 0 \qquad \text{on } \partial\Omega.$$

Hence:

$$-\int_\Omega W(x)\frac{u_\lambda^p}{(u_\lambda + \varepsilon)^p}dx = \int_\Omega p(u_\lambda + \varepsilon)^{-p-1}|\nabla u_\lambda|^2 dx + \lambda\int_\Omega \frac{u_\lambda^q}{(u_\lambda + \varepsilon)^p}dx.$$

It follows that there exists $\delta > 0$ such that:

$$\int_\Omega W(x)\frac{u_\lambda^p}{(u_\lambda + \varepsilon)^p}dx \leq -\delta < 0, \text{ for all } \varepsilon \in (0,1).$$

Letting $\varepsilon \to 0$, we have:

$$\int_\Omega W(x)dx \leq -\delta < 0.$$

Throughout, in the following, we suppose that the condition $(*)$ is satisfied.

Lemma 2.2. For all $\lambda > 0$, there exists a subsolution $\underline{u}_\lambda$, strictly positive in Ω, for the problem (1_λ).

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Proof. From [5], we know that the problem:

$$\begin{cases} -\Delta u = \lambda W(x) u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

has the first eigenvalue $\lambda_1 > 0$ and the associated first eigenfunction φ_1 is strictly positive in Ω.

Let $\varepsilon > 0$. Any $\varepsilon\varphi_1$ is a subsolution of $(\mathbf{1}_\lambda)$, provided:

$$\varepsilon\lambda_1 W(x)\varphi_1 = -\Delta(\varepsilon\varphi_1) \le \lambda\varepsilon^q\varphi_1^q + W(x)\varepsilon^p\varphi_1^p$$

which is satisfied for all $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 = \varepsilon_0(\lambda)$ small enough.

Now, we put $\underline{u}_\lambda = \varepsilon\varphi_1$ with $\varepsilon \in (0, \varepsilon_0)$ and this ends the proof.

Lemma 2.3. There exists $\Lambda_0 \in \boldsymbol{R}$, $\Lambda_0 > 0$, such that, for every $\lambda \in (0, \Lambda_0)$, the problem $(\mathbf{1}_\lambda)$ has a supersolution $\bar{u}_\lambda$.

Proof. We observe that, since $\int_\Omega W(x)\,dx < 0$, there exists $\delta > 0$ such that:

$$\int_\Omega W^+(x)\,dx < \left(\frac{1}{1+\delta}\right)^p \int_\Omega W^-(x)\,dx,$$

where:

$$W^+(x) = \max\{W(x), 0\},\ W^-(x) = \max\{-W(x), 0\},\ x \in \Omega.$$

Let $m = \left[\frac{2+\delta}{\delta}\right] + 1$, where $\left[\frac{2+\delta}{\delta}\right] = \max\left\{n \in Z : n \le \frac{2+\delta}{\delta}\right\}$, and let:

$$E_k = \left\{v \in \boldsymbol{C}^1(\bar{\Omega}) : \int_\Omega v\,dx = 0,\ \|v\|_\infty \le \frac{k}{m}\right\},$$

where $k \in \boldsymbol{R}$, $k > 0$. Denote by $H_\lambda(x, v)$ the quantity:

$$H_\lambda(x, v) = \lambda v|v|^{q-1} + W(x)v|v|^{p-1} - \frac{1}{vol\,\Omega}\int_\Omega (\lambda|v|^q + W(x)|v|^p)\,dx,\ x \in \Omega.$$

Observe that if $v \in E_k$ then $H_\lambda(x, k+v) \in \boldsymbol{C}^1(\bar{\Omega})$, since $k + v > 0$ on $\bar{\Omega}$, and:

$$|H_\lambda(x, k+v)| \le \lambda k^q\left(1+\frac{1}{m}\right)^q + 2\|W\|_\infty k^p\left(1+\frac{1}{m}\right)^p,$$

for every $x \in \Omega$ and $v \in E_k$. It is well know, since $H_\lambda(x, k+v) \in \boldsymbol{C}^1(\bar{\Omega})$ for $v \in E_k$ and since $\int_\Omega H_\lambda(x, k+v)\,dx = 0$, that the problem:

$$\begin{cases} -\Delta f = H_\lambda(x, k+v) & \text{in } \Omega \\ \dfrac{\partial f}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

is solvable (see, for example, [3], Teorema 7.1., pp. 76-78) and there exists a unique solution $f \in \boldsymbol{C}^2(\Omega) \cap \boldsymbol{C}^1(\bar{\Omega})$ verifying: $\int_\Omega f\,dx = 0$. For this solution a priori bounds are available. In fact, for all $r > 1$, there exists a constant $c_r > 0$, independent of λ, such that:

$$\|f\|_{W^{2,r}(\Omega)} \le c_r\|H_\lambda(x, k+v)\|_{L^r(\Omega)}.$$

Then, for $r > N$, it follows that:

$$\|f\|_\infty \le c_2\|H_\lambda(x, k+v)\|_{L^r(\Omega)} \le c_2(vol\,\Omega)^{\frac{1}{r}}\left[\lambda k^q\left(1+\frac{1}{m}\right)^q + 2\|W\|_\infty k^p\left(1+\frac{1}{m}\right)^p\right],$$

where c_2 is a positive constant which is independent of λ.

Observe that there exist λ_0, $k_0 > 0$ such that: $\|f\|_\infty \le \frac{k_0}{m}$, for all $\lambda \in (0, \lambda_0)$. Hence the application: $v \to f$ is well-defined and maps the convex closed set E_{k_0} into a precompact subset of E_{k_0}. By Schauder's theorem, we obtain a function $v^* \in E_{k_0}$ such that:

$$\begin{cases} -\Delta v^* = H_\lambda(x, k_0 + v^*) & \text{in } \Omega \\ \dfrac{\partial v^*}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $\bar{u}_\lambda = v^* + k_0$. We have that:

$$k_0\left(1+\frac{1}{m}\right) \ge \bar{u}_\lambda \ge k_0\left(1-\frac{1}{m}\right)$$

for every $\lambda \in (0, \lambda_0)$. Observe that:

$$-\Delta\bar{u}_\lambda - \lambda\bar{u}_\lambda^q - W(x)\bar{u}_\lambda^p = -\frac{1}{vol\,\Omega}\int_\Omega \lambda\bar{u}_\lambda^q + W(x)\bar{u}_\lambda^p\,dx.$$

Now, we prove that:

$$\int_\Omega W(x)\bar{u}_\lambda^p\,dx < 0,$$

for all $\lambda \in (0, \lambda_0)$. We have:

$$\int_\Omega W^+(x)\bar{u}_0^p\,dx \le k_0^p\left(1+\frac{1}{m}\right)^p\int_\Omega W^+(x)\,dx$$

$$< \frac{k_0^p}{(1+\delta)^p}\left(1+\frac{1}{m}\right)^p\int_\Omega W^-(x)\,dx$$

$$= \left(\frac{1}{1+\delta}\right)^p\left(\frac{1+\frac{1}{m}}{1-\frac{1}{m}}\right)^p k_0^p\left(1-\frac{1}{m}\right)^p\int_\Omega W^-(x)\,dx$$

$$\le \left(\frac{1}{1+\delta}\right)^p\left(\frac{1+\frac{1}{m}}{1-\frac{1}{m}}\right)^p\int_\Omega W^-(x)\bar{u}_\lambda^p\,dx.$$

Since $\left(\frac{1}{1+\delta}\right)^p \left(\frac{1+\frac{1}{m}}{1-\frac{1}{m}}\right)^p < 1$, by the definition of m, we obtain that:

$$\int_\Omega W(x)\bar{u}_\lambda^p dx < 0.$$

As a consequence, we can find $\lambda_0' > 0$ such that, for all $\lambda \in (0, \lambda_0')$, we have:

$$\int_\Omega (\lambda \bar{u}_\lambda^q + W(x)\bar{u}_\lambda^p) dx \leq 0.$$

Put $\Lambda_0 = \min\{\lambda_0, \lambda_0'\}$ and we observe that, for every $\lambda \in (0, \Lambda_0)$, the problem $(\mathbf{1}_\lambda)$ has a supersolution $\bar{u}_\lambda$ such that:

$$k_0\left(1+\frac{1}{m}\right) \geq \|\bar{u}_\lambda\|_\infty \geq k_0\left(1-\frac{1}{m}\right).$$

Proof of Theorem 1.1. Let $\lambda \in (0, \Lambda_0)$. Clearly, from the proofs of Lemmas 2.2. and 2.3., there exists a subsolution $\underline{u}_\lambda$ and a supersolution $\bar{u}_\lambda$, for the problem $(\mathbf{1}_\lambda)$, such that $\underline{u}_\lambda \leq \bar{u}_\lambda$. From the result of Sattinger (see [6]), we obtain a solution u_λ for $(\mathbf{1}_\lambda)$ such that $\underline{u}_\lambda \leq u_\lambda \leq \bar{u}_\lambda$ in $\bar{\Omega}$. To complete the proof, it remains to show that $\|u_\lambda\|_\infty \to 0$ as $\lambda \downarrow 0$. But, from the proof of Lemma 2.3., we observe that $\|u_\lambda\|_\infty \leq \|\bar{u}_\lambda\|_\infty \leq k_0\left(1+\frac{1}{m}\right)$, for every $\lambda \in (0, \Lambda_0)$. Clearly, following the arguments used in this proof, for $\lambda_0 > 0$ sufficiently small, we can choose $k_0 > 0$ arbitrary small. This completes the proof.

Denote by Λ the quantity:

$$\Lambda = \sup\{\lambda > 0 : (\mathbf{1}_\lambda) \text{ has solution}\}.$$

Clearly: $\Lambda \geq \Lambda_0 > 0$.

Proposition 2.1. For all $\lambda \in (0, \Lambda)$ the problem $(\mathbf{1}_\lambda)$ has a solution.

Proof. This proof is inspired by the proof of Lemma 3.2. in [1]. Let $0 < \lambda < \Lambda$ and let $\mu \in (\lambda, \Lambda)$ such that u_μ is a solution of $(\mathbf{1}_\mu)$. It is easy to show that u_μ is a supersolution for $(\mathbf{1}_\lambda)$. Choosing $\varepsilon > 0$ sufficiently small, we have that $\varepsilon\varphi < u_\mu$ and, from the results of Sattinger (see [6]), it follows that $(\mathbf{1}_\lambda)$ has a solution.

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