A Decomposition of R-polynomials and Kazhdan-Lusztig Polynomials

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(Communicated by Shokichi IYANAGA, M. J. A., June 13, 1995)

The R-polynomial is defined for two elements in an arbitrary Coxeter group. These polynomials are intimately related to Kazhdan-Lusztig polynomials introduced by Kazhdan and Lusztig in 1979 ([4]). For example, it is well known that

$$q^{l(w)-l(x)}P_{x,w}\left(\frac{1}{q}\right) = \sum_{x\le y\le w} R_{x,y}(q)P_{y,w}(q),$$

where $P_{x,w}(q)$ (resp. $R_{x,w}(q)$) is the Kazhdan-Lusztig polynomial (resp. the R-polynomial).

In [2], F. Brenti found a decomposition formula of R-polynomials for symmetric groups and he showed that products of R-polynomials for symmetric groups are also R-polynomials for symmetric groups. The purpose of this article is to find a decomposition formula of R-polynomials and Kazhdan-Lusztig polynomials for arbitrary Coxeter groups in extension of Brenti's result.

First, we recall the definition of the Bruhat order and R-polynomials. Throughout this article, (W, S) is an arbitrary Coxeter system, where S denotes a privileged set of involutions in W. The standard references are [1] and [3] for the Bruhat order and R-polynomials.

Definition (Bruhat order). We put $T := \{wsw^{-1}; s \in S, w \in W\}$. For $y, z \in W$, we denote $y <' z$ if and only if there exists an element t of T such that $l(tz) < l(z)$ and $y = tz$, where l is the length function. Then the Bruhat order denoted by $\le$ is defined as follows. For x, $w \in W$, $x \le w$ if and only if there exists a sequence $x_0, x_1, \dots, x_r$ in W such that $x = x_0 <' x_1 <' \cdots <' x_r = w$.

The following is well known. For $w \in W$, let $s_1 s_2 \cdots s_m$ be a reduced expression of w, i.e. $w = s_1 s_2 \cdots s_m$, $s_i \in S$ for all $i \in [m] (:= \{1,2,\dots, m\})$ and $l(w) = m$. For $x \in W$, $x \le w$ if and only if there exists a sequence of natural numbers $i_1, i_2, \dots, i_t$ such that $1 \le i_1 < i_2 < \cdots < i_t \le m$ and $x = s_{i_1}s_{i_2}\cdots s_{i_t}$. This expression of x is not reduced in general, i.e. it may happen that $l(x) < t$. However it is known that one can find a sequence of natural numbers $j_1, j_2, \dots, j_k$ such that $1 \le j_1 < j_2 < \cdots < j_k \le m$, $x = s_{j_1}s_{j_2}\cdots s_{j_k}$ and $l(x) = k$.

Also, the following decomposition called the coset decomposition is well known. Let J be a subset of S. We put $W_J :=$ subgroup of W generated by J and $W^J := \{y \in W\,; l(yz) = l(y) + l(z)$ for any $z \in W_J\}$. Then, for $w \in W$, there uniquely exist $w^J \in W^J$ and $w_J \in W_J$ such that $w = w^J w_J$, whence follows:

Lemma A. *Let $y, z \in W$. If $G(y) \cap G(z) = \phi$, where $G(y) := \{s \in S\,; s \le y\}$, then we have $l(yz) = l(y) + l(z)$.*

R-polynomials are defined as follows:

Definition-Proposition (R-polynomial). *$\mathcal{H}(W)$ is the Hecke algebra associated to W. That is, $\mathcal{H}(W)$ is the free $\mathbf{Z}[q, q^{-1}]$-module having the set $\{T_w; w \in W\}$ as a basis with the multiplication such that*

$$T_w T_s = \begin{cases} T_{ws} & \text{if } l(ws) > l(w), \\ qT_{ws} + (q-1)T_w & \text{if } l(ws) < l(w) \end{cases}$$

for all $w \in W$ and $s \in S$. For $w \in W$, there exists a unique family of polynomials $\{R_{x,w}(q)\}_{x \le w} \subset \mathbf{Z}[q]$ satisfying

$$(T_{w^{-1}})^{-1} = q^{-l(w)} \sum_{x \le w} (-1)^{l(w)-l(x)} R_{x,w}(q) T_x.$$

We put $R_{x,w}(q) := 0$ if $x \not\le w$ for convenience. $R_{x,w}(q)$ is called the R-polynomial for x, $w \in W$.

By using R-polynomials, we can define Kazhdan-Lusztig polynomials as follows:

Definition-Proposition (Kazhdan-Lusztig polynomial). *There exists a unique family of polynomials $\{P_{x,w}(q)\}_{x,w\in W} \subset \mathbf{Z}[q]$ satisfying the following conditions:*

(i) $P_{x,w}(q) = 0$ *if* $x \not\le w$,

(ii) $P_{x,x}(q) = 1$,

(iii) $\deg P_{x,w}(q) \le \frac{1}{2}(l(w) - l(x) - 1)$ *if* $x < w$,

(iv) $q^{l(w)-l(x)}P_{x,w}\left(\frac{1}{q}\right) = \sum_{x\le y\le w} R_{x,y}(q)P_{y,w}(q)$ *if* $x \le w$.

$P_{x,w}(q)$ is called the Kazhdan-Lusztig polynomial

for x, $w \in W$.

Our main result is the following.

Theorem A. *Let* $x_1, x_2, w_1, w_2 \in W$ *with* $x_1 \leq w_1$ *and* $x_2 \leq w_2$. *If* $G(w_1) \cap G(w_2) = \phi$, *then we have*

(i) $R_{x_1,w_1}(q)R_{x_2,w_2}(q) = R_{x_1x_2,w_1w_2}(q)$,

(ii) $P_{x_1,w_1}(q)P_{x_2,w_2}(q) = P_{x_1x_2,w_1w_2}(q)$.

Proof. (i) Let $s_1s_2\cdots s_r$ (resp. $s_{r+1}s_{r+2}\cdots s_m$) be a reduced expression of w_1 (resp. w_2). Note that $s_1s_2\cdots s_m$ is a reduced expression of w_1w_2 by Lemma A. So, by the definition of R-polynomials and the fact that $(T_{(w_1w_2)^{-1}})^{-1} = (T_{w_1^{-1}})^{-1}(T_{w_2^{-1}})^{-1}$, we have

$$\sum_{x\leq w_1w_2}(-1)^{l(w_1w_2)-l(x)}R_{x,w_1w_2}(q)T_x$$
$$=\sum_{x_1'\leq w_1}(-1)^{l(w_1)-l(x_1')}R_{x_1',w_1}(q)T_{x_1'}\sum_{x_2'\leq w_2}(-1)^{l(w_2)-l(x_2')}R_{x_2',w_2}(q)T_{x_2'}$$
$$=\sum_{x_1'\leq w_1,x_2'\leq w_2}(-1)^{l(w_1w_2)-l(x_1'x_2')}R_{x_1',w_1}(q)R_{x_2',w_2}(q)T_{x_1'x_2'}.$$

We suppose that there exist $x_1' \leq w_1$ and $x_2' \leq w_2$ such that $T_{x_1x_2} = T_{x_1'x_2'}$. Then, we have $x_1x_2 = x_1'x_2'$. We put $J := S\setminus G(w_1)$ and then we can easily check that $x_1, x_1' \in W^J$ and $x_2, x_2' \in W_J$. Hence, by the uniqueness of the coset decomposition, we see that $x_1' = x_1$ and $x_2' = x_2$. So, the coefficient of $T_{x_1x_2}$ in the right hand side is equal to $(-1)^{l(w_1w_2)-l(x_1x_2)}R_{x_1,w_1}(q)R_{x_2,w_2}(q)$. Hence, it turns out that

$$R_{x_1,w_1}(q)R_{x_2,w_2}(q) = R_{x_1x_2,w_1w_2}(q).$$

(ii) We will show (ii) by induction on $l(w_1w_2) - l(x_1x_2)$. In case $l(w_1w_2) - l(x_1x_2) = 0$, then we see that $x_1x_2 = w_1w_2$, $x_1 = w_1$ and $x_2 = w_2$. So, we have $P_{x_1,w_1}(q)P_{x_2,w_2}(q) = 1 = P_{x_1x_2,w_1w_2}(q)$. We suppose that (ii) is correct up to the case where $l(w_1w_2) - l(x_1x_2) = k-1$ $(k \geq 1)$ and we will show (ii) in case $l(w_1w_2) - l(x_1x_2) = k$. By Definition-Proposition (Kazhdan-Lusztig polynomial)-(iv) and our inductive hypothesis, we have

$$q^{l(w_1w_2)-l(x_1x_2)}P_{x_1x_2,w_1w_2}\left(\frac{1}{q}\right)$$
$$=\sum_{x_1x_2\leq y\leq w_1w_2}R_{x_1x_2,y}(q)P_{y,w_1w_2}(q)$$
$$=\sum_{x_1\leq y_1\leq w_1,x_2\leq y_2\leq w_2}R_{x_1x_2,y_1y_2}(q)P_{y_1y_2,w_1w_2}(q)$$
$$=\sum_{x_1<y_1\leq w_1,x_2<y_2\leq w_2}R_{x_1,y_1}(q)R_{x_2,y_2}(q)P_{y_1,w_1}(q)P_{y_2,w_2}(q)+P_{x_1x_2,w_1w_2}(q)$$
$$+\sum_{x_1<y_1\leq w_1}R_{x_1,y_1}(q)P_{y_1,w_1}(q)P_{x_2,w_2}(q)$$
$$+\sum_{x_2<y_2\leq w_2}R_{x_2,y_2}(q)P_{x_1,w_1}(q)P_{y_2,w_2}(q)$$
$$=q^{l(w_1w_2)-l(x_1x_2)}P_{x_1,w_1}\left(\frac{1}{q}\right)P_{x_2,w_2}\left(\frac{1}{q}\right)-P_{x_1,w_1}(q)P_{x_2,w_2}(q)+P_{x_1x_2,w_1w_2}(q).$$

Hence, we have

$$q^{l(w_1w_2)-l(x_1x_2)}P_{x_1x_2,w_1w_2}\left(\frac{1}{q}\right)-P_{x_1x_2,w_1w_2}(q)$$
$$=q^{l(w_1w_2)-l(x_1x_2)}P_{x_1,w_1}\left(\frac{1}{q}\right)P_{x_2,w_2}\left(\frac{1}{q}\right)-P_{x_1,w_1}(q)P_{x_2,w_2}(q).$$

Note that $\deg P_{x_1x_2,w_1w_2}(q) \leq \frac{1}{2}(l(w_1w_2) - l(x_1x_2) - 1)$ and $\deg P_{x_1,w_1}(q) + \deg P_{x_2,w_2}(q) \leq \frac{1}{2}(l(w_1w_2) - l(x_1x_2) - 1)$. It follows that

$$P_{x_1x_2,w_1w_2}(q) = P_{x_1,w_1}(q)P_{x_2,w_2}(q).$$

Thus we have shown (ii) by induction. □

By Theorem A, we can easily obtain the following.

Corollary A. *Let* $S_1, S_2, \ldots, S_k$ *be subsets of* S *satisfying* $S_i \cap S_j = \phi$ *for all* $i \neq j$ *in* $[k]$. *Then, we have*

$$R(W_{S_1})R(W_{S_2})\cdots R(W_{S_k}) \subset R(W),$$
$$P(W_{S_1})P(W_{S_2})\cdots P(W_{S_k}) \subset P(W),$$

where $R(G)$ (*resp.* $P(G)$) *is the set of* R-*polynomials* (*resp. Kazhdan-Lusztig polynomials*) *for a Coxeter group* G.

For example, let $W(A_n)$, $W(B_m)$ and $W(D_r)$ be Weyl groups of type A_n, B_m and D_r respectively, then we have

$$R(W(A_n))R(W(A_m)) \subset R(W(A_{n+m})),$$
$$P(W(A_n))P(W(A_m)) \subset P(W(A_{n+m})),$$
$$R(W(A_n))R(W(B_m)) \subset R(W(B_{n+m})),$$
$$P(W(A_n))P(W(B_m)) \subset P(W(B_{n+m})),$$
$$R(W(A_n))R(W(D_m)) \subset R(W(D_{n+m})),$$
$$P(W(A_n))P(W(D_m)) \subset P(W(D_{n+m})).$$

References

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