16. *Singular Variation of Domains and L^∞ Boundedness of Eigenfunctions for some Semi-linear Elliptic Equations*

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1. Introduction. Let M be a bounded domain in $\boldsymbol{R}^2$ with smooth boundary ∂M. Let w be a fixed point in M. By $B(\varepsilon\,;w)$ we denote the ball of center w with radius $\varepsilon > 0$. We remove $\overline{B(\varepsilon\,;w)}$ from M and we put $M_\varepsilon = M \setminus \overline{B(\varepsilon\,;w)}$. We write $B(\varepsilon\,;w) = B_\varepsilon$.

Fix $p \in (1, \infty)$. We put

$$\lambda(\varepsilon) = \inf_{X_\varepsilon} \int_{M_\varepsilon} |\nabla u|^2\, dx, \tag{1.1}$$

where $X_\varepsilon = \{u \in H^1(M_\varepsilon) : \| u \|_{L^{p+1}(M_\varepsilon)} = 1,\ u = 0 \text{ on } \partial M,\ u \geq 0 \text{ in } M_\varepsilon\}$. Then, we know that there exists at least one solution u_ε which attains (1.1). It satisfies

$$\begin{aligned} -\Delta u_\varepsilon &= \lambda(\varepsilon) u_\varepsilon^p && \text{in } M_\varepsilon, \\ \frac{\partial}{\partial \nu_x} u_\varepsilon &= 0 && \text{on } \partial B_\varepsilon, \\ u_\varepsilon &= 0 && \text{on } \partial M. \end{aligned} \tag{1.2}$$

Here $\partial / \partial \nu_X$ denotes the exterior normal derivative.

In this paper we prove the following Theorem 1.

Theorem 1. *There exists a positive constant C independent of ε such that*

$$\sup_{u_\varepsilon \in S_\varepsilon} \sup_{x \in M_\varepsilon} u_\varepsilon(x) < C, \tag{1.3}$$

where S_ε is the set of minimizers of (1.1).

The reader may be referred to Ozawa [2],[3], Lin [1] for related problems.

2. Preliminary lemma. Lemma 2.1. *Assume that $u_\varepsilon \in C^\infty(M_\varepsilon)$ is harmonic in M_ε and $u_\varepsilon = 0$ for any $x \in \partial M$ and that*

$$\max\{| \partial u_\varepsilon(x) / \partial \nu_x | \,;\, x \in \partial B(\varepsilon\,;w)\} = L.$$

Then, $| u_\varepsilon(x) | \leq C\varepsilon L(1 + \log(| x - w | / \varepsilon))$ for any $x \in M_\varepsilon$. Here C is a positive constant independent of ε.

Lemma 2.1 is given in Ozawa [4].

Let $G_\varepsilon(x, y)$ be the Green function of the Laplacian in M_ε satisfying

$$\begin{aligned} -\Delta_x G_\varepsilon(x, y) &= \delta(x - y) && x, y \in M_\varepsilon, \\ G_\varepsilon(x, y)_{|x \in \partial M} &= 0 && y \in M_\varepsilon, \\ \frac{\partial}{\partial \nu_x} G_\varepsilon(x, y)_{|x \in \partial B_\varepsilon} &= 0 && y \in M_\varepsilon. \end{aligned}$$

Let $G(x, y)$ be the Green function of the Laplacian in M under the Dirichlet condition on ∂M. We put

$$\langle \nabla_w a(x, w), \nabla_w b(w, y) \rangle = \sum_{i=1}^{2} \frac{2}{\partial w_i} a(x, w) \frac{\partial}{\partial w_i} b(w, y).$$

Here $w = (w_1, w_2)$ is the standard orthogonal coordinates of w. We put

$$R_\varepsilon(x, y) = G(x, y) + 2\pi\varepsilon^2\langle \nabla_w G(x, w), \nabla_w G(w, y)\rangle.$$

We put

$$\boldsymbol{G}_\varepsilon f(x) = \int_{M_\varepsilon} G_\varepsilon(x, y) f(y)\,dy$$

and

$$\boldsymbol{R}_\varepsilon f(x) = \int_{M_\varepsilon} R_\varepsilon(x, y) f(y)\,dy.$$

We write $\| f \|_{L^q(M_\varepsilon)}$ as $\| f \|_{q,\varepsilon}$. We have the following

Lemma 2.2. *Fix $q > 2$. Fix $f \in L^q(M_\varepsilon)$.*
Then, there exists a constant $C > 0$ independent of ε such that

$$\max_{x \in \partial B_\varepsilon} \left| \frac{\partial}{\partial \nu_x} (\boldsymbol{R}_\varepsilon f(x) - \boldsymbol{G}_\varepsilon f(x)) \right| \le C\varepsilon^\tau \| f \|_{q,\varepsilon} \tag{2.1}$$

holds for $\tau = 1 - (2/q)$.

Proof. Since $(\partial/\partial\nu_x)\boldsymbol{G}_\varepsilon f(x) = 0$ for $x \in \partial B_\varepsilon$, we have to get bound of $(\partial/\partial\nu_x)\boldsymbol{R}_\varepsilon f$ to prove (2.1). We know that $G(x, y) + (2\pi)^{-1}\log|x - y| = S(x, y) \in C^\infty(M \times M)$. By Ozawa [4, (2.9), p. 644] we have

$$\frac{\partial}{\partial \nu_x} \boldsymbol{R}_\varepsilon f(x)\,|_{x=w+(\varepsilon,0)} = \frac{\partial}{\partial x_1} \boldsymbol{G}\hat{f}(x) - \frac{\partial}{\partial w_1} \boldsymbol{G}\hat{f}(w) + 2\pi\varepsilon^2 \frac{\partial}{\partial x_1} \langle \nabla_w S(x, w), \nabla_w(\boldsymbol{G}\hat{f})(w)\rangle. \tag{2.2}$$

Here $\hat{f}$ is an extension of f to M which is zero outside M_ε. Therefore, absolute value of the left-hand side of (3.2) does not exceed $C\varepsilon^\tau \| \boldsymbol{G}f \|_{C^{1+\tau}(\bar{M})} + O(\varepsilon^2) \| f \|_{q,\varepsilon}$ for $q > 2$. By the Sobolev embedding theorem applied to $\boldsymbol{G}f$ we get (3.1).

Proof of Theorem 1. Assume that $q > 2$. Let u_ε be the solution of (1.2). By Lemmas 2.1 and 2.2 we have

$$\begin{aligned} &| \boldsymbol{G}_\varepsilon u_\varepsilon^p(x) - \boldsymbol{R}_\varepsilon u_\varepsilon^p(x) | \\ &\le C\varepsilon^{1+\tau}(1 + \log(|x - w|/\varepsilon)) \| u_\varepsilon^p \|_{q,\varepsilon} \\ &\le C\varepsilon^{2-(2/q)} |\log\varepsilon| \, \| u_\varepsilon \|_{pq,\varepsilon}^p \end{aligned} \tag{2.3}$$

for arbitrary $x \in M_\varepsilon$. We recall that $\| u_\varepsilon \|_{p+1,\varepsilon} = 1$, $|\lambda(\varepsilon)| \le C$.

Let $\tilde{u}_\varepsilon$ be an extension of u_ε to M which is defined in the Lemma A in the appendix of this paper.
Then,

$$\begin{aligned} \| \tilde{u}_\varepsilon \|_{H^1(M)} &\le C \| u_\varepsilon \|_{H^1(M_\varepsilon)} + C\varepsilon^{-2/(p+1)} \| u_\varepsilon \|_{p+1,\varepsilon} \\ &\le C'\varepsilon^{-2/(p+1)}. \end{aligned} \tag{2.4}$$

By the Sobolev embedding $H^1(M) \hookrightarrow L^{pq}(M)$, we can see that

$$\begin{aligned} \| u_\varepsilon \|_{pq,\varepsilon} &\le \| \tilde{u}_\varepsilon \|_{L^{pq}(M)} \\ &\le C \| \tilde{u}_\varepsilon \|_{H^1(M)} \\ &\le C'\varepsilon^{-2/(p+1)}. \end{aligned}$$

We take $q > p + 1$. Then, by (2.3) and (2.4), we get

$$\begin{aligned} | \boldsymbol{G}_\varepsilon u_\varepsilon^p(x) - \boldsymbol{R}_\varepsilon u_\varepsilon^p(x) | &\le C\varepsilon^{(2/(p+1))-(2/q)} |\log\varepsilon| \\ &\le C' < \infty, \end{aligned} \tag{2.5}$$

for any $x \in M_\varepsilon$.

On the other hand, by using the smoothing property of the operator G, we have for any $x \in \partial M_\varepsilon$

$$(2.6)\quad \begin{aligned} |\boldsymbol{R}_\varepsilon u_\varepsilon^p(x)| &\leq |\boldsymbol{G}\hat{u}_\varepsilon^p(x)| + 2\pi\varepsilon^2 |\langle \nabla_w G(x, w), \nabla_w(\boldsymbol{G}\hat{u}_\varepsilon^p)(w)\rangle| \\ &\leq C\| u_\varepsilon \|_{p+1,\varepsilon}^p \\ &\quad + C\varepsilon\Big(\int_{M_\varepsilon} |\nabla_w G(w, y)|^{p+1} dy\Big)^{1/(p+1)} \| u_\varepsilon \|_{p+1,\varepsilon}^p \\ &\leq C(1 + \varepsilon^{2/(p+1)}) \leq C'. \end{aligned}$$

Here $\hat{f}$ denotes the extension of f to M as zero outside M_ε.

From (2.5) and (2.6) we can see that $|u_\varepsilon(x)| \leq C$ for $x \in M_\varepsilon$ by using $u_\varepsilon = \lambda(\varepsilon)\boldsymbol{G}_\varepsilon u_\varepsilon^p$. Now our proof of Theorem 1 is complete.

Appendix. Lemma A. *There exists an extension operator* $E : H^1(M_\varepsilon) \ni u \mapsto Eu = \tilde{u} \in H^1(M)$ *satisfying the following*:

(0) E *is linear.*

(1) $\tilde{u}(x) = u(x)$, M_ε
holds for any $u \in H^1(M_\varepsilon)$.

(2) $\|\tilde{u}\|_{L^s(M)} \leq C\|u\|_{L^s(M_\varepsilon)} \quad (1 \leq s \leq \infty)$
holds for any $u \in H^1(M_\varepsilon) \cap L^s(M_\varepsilon)$.

(3) $\|\tilde{u}\|_{H^1(M)} \leq C\|u\|_{H^1(M_\varepsilon)} + C\varepsilon^{-2/s}\|u\|_{L^s(M_\varepsilon)}$
holds for any $u \in H^1(M_\varepsilon) \cap L^s(M_\varepsilon)$ *with* $2 \leq s < \infty$.

Proof. Without loss of generality, we may assume that $w = 0$. We take an arbitrary $u \in H^1(M_\varepsilon)$ and put

$$\begin{aligned} \tilde{u}(x) &= u(x) \qquad x \in M_\varepsilon \\ &= u(\varepsilon^2 x |x|^{-2})\eta_\varepsilon(x) \qquad x = \overline{B_\varepsilon}, \end{aligned}$$

where $\eta_\varepsilon \in C^\infty(\boldsymbol{R}^2)$ satisfies $0 \leq \eta_\varepsilon \leq 1$, $\eta_\varepsilon = 1$ on $\boldsymbol{R}^2 \setminus \overline{B}_{\varepsilon/2}$, $\eta_\varepsilon = 0$ on $B_{\varepsilon/4}$ and $|\nabla\eta_\varepsilon| \leq 8\varepsilon^{-1}$. Notice that both $\eta_\varepsilon(\varepsilon^2 x|x|^{-2})$ and $(\nabla\eta_\varepsilon)(\varepsilon^2 x|x|^{-2})$ vanish on $\boldsymbol{R}^2 \setminus B_{4\varepsilon}$. Then, by using the Kelvin transformation of co-ordinates $y = \varepsilon^2 x|x|^{-2}$, we have

$$\begin{aligned} \int_{B_\varepsilon} |\tilde{u}(x)|^s dx &= \int_{\boldsymbol{R}^2 \setminus \bar{B}_\varepsilon} |u(y)|^s \eta_\varepsilon(\varepsilon^2 y|y|^{-2})^s (\varepsilon|y|^{-1})^4 \, \mathrm{dy} \\ &\leq \int_{M_\varepsilon} |u(y)|^s dy \qquad (1 \leq s \leq \infty), \end{aligned}$$

where the factor $(\varepsilon|y|^{-1})^4$ comes from the absolute value of the Jacobian determinant of the Kelvin transformation. We also have

$$\begin{aligned} \int_{B_\varepsilon} |\nabla\tilde{u}(x)|^2 dx &\leq C\int_{B_\varepsilon} |u(\varepsilon^2 x|x|^{-2})|^2 |(\nabla\eta_\varepsilon)(x)|^2 dx \\ &\quad + C\int_{B_\varepsilon} (\varepsilon|x|^{-1})^4 |(\nabla u)(\varepsilon^2 x|x|^{-2})|^2 \eta_\varepsilon(x)^2 dx \\ &\leq C\varepsilon^2 \int_{M_\varepsilon} |u(y)|^2 |y|^{-4} dy + C\int_{M\varepsilon} |\nabla u|^2 dy. \end{aligned}$$

By Hölder's inequality, we see that

$$\int_{M_\varepsilon} |u(y)|^2 |y|^{-4} dy \leq C\varepsilon^{-(1+(2/s))2} \|u\|_{L^s(M_\varepsilon)}^2 \quad (2 \leq s < \infty).$$

Thus, we prove Lemma A.

References

[1] S. S. Lin: Semilinear elliptic equations on singularly pertubed domains. Comm. Partial Differential Equations, **16**, 617–645 (1991).

[2] S. Ozawa: Singular variation of the ground state eigenvalue for a semilinear elliptic equation. Tôhoku Math. J., **45**, 359–368 (1993).

[3] ——: L^∞ boundedness of nonlinear eigenfunction under singular variation of domains (preprint).

[4] ——: Asymptotic property of an eigenfunction of the Laplacian under singular variation of domains-The Neumann condition. Osaka J. Math., **22**, 639–655 (1985).