60. *An Application of Frey's Idea to Exponential Diophantine Equations*

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In 1956 Sierpiński ([6]) proved that the equation $3^x + 4^y = 5^z$ has no solutions in natural numbers x, y, z except one, namely, $x = y = z = 2$. Jeśmanowicz ([5]) soon followed him by proving that the only solution of each of the equations

$$5^x + 12^y = 13^z,\ 7^x + 24^y = 25^z,\ 9^x + 40^y = 41^z,\ 11^x + 60^y = 61^z$$

in natural numbers x, y, z is also $x = y = z = 2$, and asked whether there exists a Pythagorean triple (a, b, c) for which the equation $a^x + b^y = c^z$ has a solution in natural numbers x, y, z different from $x = y = z = 2$ (cf. also [8]-[12]).

In the present paper, applying Frey's idea ([3]) (which reduces Fermat's problem to the existence of a certain kind of elliptic curve) to Jeśmanowicz's problem, we will give an effective procedure to determine whether there are no other solutions than $x = y = z = 2$ for a given Pythagorean triple.

This method, in fact, has much broader application as follows:

Theorem. *Let a, b, c, l, m, n be given relatively prime natural numbers. Then the equation*

$$la^x + mb^y = nc^z \tag{1}$$

has finitely many solutions, all of which can be effectively determined.

In what follows we will give a proof of this theorem and a few examples.

The following theorem is useful in reducing calculation:

Frey's theorem ([3]). *For relatively prime integers a, b, c and natural numbers n_1, n_2, n_3 suppose that (n_1, n_2, n_3) is a solution of the equation* (1) *with $la^{n_1} \equiv \pm 1 \pmod 4$ and $mb^{n_2} \equiv 0 \pmod{2^4}$. Let E be the elliptic curve defined by the equation*

$$y^2 = x(x - la^{n_1})(x + mb^{n_2}). \tag{2}$$

Then the curve E is stable with discriminant

$$(lmna^{n_1}b^{n_2}c^{n_3}/16)^2$$

and conductor

$$\prod_{p \mid lmnab^{n_2}c/16} p.$$

Now, let (a, b, c) be a primitive Pythagorean triple with b even (and hence divisible by 4), and suppose that there exist natural numbers n_1, n_2, n_3 which satisfy the equation

$$a^{n_1} + b^{n_2} = c^{n_3}. \tag{3}$$

Next, define the elliptic curve E by the equation

$$y^2 = x(x - a^{n_1})(x + b^{n_2}).$$

The discriminant Δ and the conductor N of the curve E are

$$\Delta=(a^{n_1}b^{n_2}c^{n_3}/16)^2,\quad N=\prod_{p\mid ab^{n_2}c/16} p,$$

respectively, by Frey's theorem.

The following theorem is fundamental for our purpose:

Shafarevich–Coates' theorem. *Let S be a finite set of prime numbers. Then, up to isomorphism over $\boldsymbol{Q}$, there are only a finite number of elliptic curves over $\boldsymbol{Q}$ having good reduction at all primes not in S. Moreover, there is an effective procedure for determining all such elliptic curves.*

For the proof of the first part, for example, see [7], pp. 263-264 or [4], pp. 287-288. The latter part was proved in [1].

The only thing we have to do is to check whether the relation (3) holds for exponents of each discriminant of the elliptic curves thus determined.

Example 1 (the case where $a=3$, $b=4$, $c=5$). We assume that there exist natural numbers n_1, n_2, n_3 satisfying the equality

$$3^{n_1}+4^{n_2}=5^{n_3}.$$

We define an elliptic curve by the equation

$$y^2=x(x-3^{n_1})(x+4^{n_2}).$$

First we suppose that $n_2\geq 2$. Then the conductor N is 15 or 30 according as $n_2=2$ or $n_2>2$. By [2], we know all elliptic curves with conductor 15 or 30. We should note that elliptic curves with conductor 15 or 30 are all modular, since the conductor is small enough. In what follows we list the elliptic curves with discriminant of a perfect square number among them:

case $N=15$: $3^4 5^4$, $3^8 5^2$, $3^2 5^2$

case $N=30$: $2^2 3^6 5^2$, $2^6 3^2 5^6$.

It is easy to check that none of the exponents (n_1, n_2, n_3) of these discriminants satisfy the equation $3^{n_1}+4^{n_2}=5^{n_3}$ except the case $n_1=n_2=n_3=2$ which corresponds to the discriminant $3^4 5^4$.

Next we deal with the case $n_2=1$. In this case the elliptic curve E defined by $y^2=x(x-3^{n_1})(x+4)$ has the discriminant

$$(3^{n_1}\cdot 4\cdot 5^{n_3})^2 \tag{4}$$

and is stable at any prime but 2. Therefore, its conductor takes the form $2^f\cdot 3\cdot 5$, where $f\leq 4$, because it is known that, in general, the discriminant is divisible by the conductor. In the same way as above we can easily check by [2] that there is no such elliptic curve with discriminant (4) whose exponents satisfy the equation $3^{n_1}+4=5^{n_3}$. This provides another proof of Sierpiński's theorem.

It is clear that the same procedure as above works in a few more case with small a, b, c. And it is also clear that, by considering the elliptic curve defined by the equation

$$y^2=x(x-la^{n_1})(x+mb^{n_2}),$$

the equation (1) has a finite number of solutions in natural numbers, all of which can be effectively determined, because the elliptic curve has good reduction at any prime which does not divide $lmnabc$.

Example 2. We will determine the solutions of the equation

$$3^x=2^y+1. \tag{5}$$

Suppose that $(x, y) = (n_1, n_2)$ is a solution in natural numbers of (5). Since it is clear that in case $n_2 \leq 4$, we have no solutions other than (1,1) and (2,3), we suppose $n_2 \geq 5$.

Consider the elliptic curve defined by the equation

$$y^2 = x(x - 3^{n_1})(x + 2^{n_2}),$$

which is stable, since $n_2 \geq 5$. Hence, by Frey's theorem, we see that the conductor of this curve is 6. But there are no elliptic curves with conductor 6 (cf. [2]). Hence the equation (5) does not have any solutions such that $y \geq 5$. Thus we learn that the equation (5) has the only solution $x = 2$, $y = 3$ in natural numbers each greater than 1.

Remark. The equation

$$3^2 = 2^3 + 1$$

corresponds to the elliptic curve

$$y^2 = x(x - 9)(x + 8)$$

with conductor $2^3 \cdot 3$ and discriminant $2^{10} \cdot 3^4$. And the equation

$$3 = 2 + 1$$

corresponds to the elliptic curve

$$y^2 = x(x - 3)(x - 2)$$

with conductor $2^5 \cdot 3$ and discriminant $2^6 \cdot 3^2$.

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