54. *On the Logarithmic Gradient of Poincaré Metric*

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1. Introduction. Let $\mathscr{D} \subset \mathscr{C}$ be a simply connected domain with at least two boundary points and let $f(z)$ be a conformal mapping of $\mathscr{B} = \{z : |z| < 1\}$ onto $\mathscr{D}$. The poincaré metric of $\mathscr{D}$ is defined by

(1) $$\lambda_{\mathscr{D}}(f(z))\,|f'(z)| = \lambda_{\mathscr{B}}(z) = 1/(1-|z|^2), \quad z \in \mathscr{B}.$$

This definition is independent of the choice of conformal mapping and because of this convenient choices available. Namely, let $w \in \mathscr{D}$ and choose the conformal mapping so that $f(0) = w$. Then

(2) $$\lambda_{\mathscr{D}}(w) = 1/|f'(0)|.$$

If $f(z)$ is a conformal mapping of a domain $\mathscr{G}$ onto $\mathscr{D}$ then, from (1) and (2), we have

(3) $$\lambda_{\mathscr{D}}(f(z))\,|f'(z)| = \lambda_{\mathscr{G}}(z), \quad z \in \mathscr{G}.$$

Given $z \in \mathscr{D}$, let $d(z, \partial\mathscr{D})$ denote the distance from z to $\partial\mathscr{D}$, it is well-known that

(4) $$1/4 \leqq d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z) \leqq 1, \quad z \in \mathscr{D}.$$

Osgood proved in [1] the following

Theorem A. *If $\mathscr{D} \subset \mathscr{C}$ is simply connected and if f is analytic and univalent in $\mathscr{D}$ then*

(5) $$|f''(z)/f'(z)| \leqq 8\lambda_{\mathscr{D}}(z)$$

for all $z \in \mathscr{D}$. The inequality is sharp.

Theorem B. *If $\mathscr{D}$ is a proper subdomain of $\mathscr{C}$ and if f is analytic and univalent in $\mathscr{D}$ then*

(6) $$|f''(z)/f'(z)| \leqq 4/d(z, \partial\mathscr{D}), \quad z \in \mathscr{D}.$$

The inequality is sharp in the sense that the equality holds for $\mathscr{D} = \mathscr{B}$ and $f(z) = z/(1-z)^2$.

Our Theorem 1 generalizes the above Theorems A and B, which reveals the relationship of (5) and (6).

Theorem 1. *If $\mathscr{D} \subset \mathscr{C}$ is simply connected domain with at least two boundary points and $f(z)$ is analytic and univalent in $\mathscr{D}$ then*

(7) $$\left|\frac{f''(z)}{f'(z)}\right| \leqq \frac{4}{d(z, \partial\mathscr{D})}\left(4\sqrt{d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z)} - 2d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z) - 1\right)$$

for all $z \in \mathscr{D}$, and the inequality is sharp.

For a differentiable function u we shall use the familiar operator

$$u_z = (u_x - iu_y)/2, \quad z = x + iy.$$

If $w = f(z)$ is a conformal mapping of a domain $\mathscr{G}$ onto $\mathscr{D}$ then from (3)

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(8) $$\log \lambda_{\mathscr{D}} \circ f + \log |f'| = \log \lambda_{\mathscr{G}}.$$

Differentiating with respect to z and using the chain rule leads to

(9) $$((\log \lambda_{\mathscr{D}})_w \circ f) f' + f''/2f' = (\log \lambda_{\mathscr{G}})_z.$$

This relation also holds in the case when $\mathscr{G} = \mathscr{B}$ and f is analytic covering map onto a multiply connected domain. When $\mathscr{D}$ is simply connected, Osgood also estimated for $|\nabla \log \lambda_{\mathscr{D}}| = 2|(\log \lambda_{\mathscr{D}})_z|$ and obtained the following

Theorem C. *If $\mathscr{D} \subset \mathscr{C}$ is simply connected then*

(10) $$|\nabla \log \lambda_{\mathscr{D}}(z)| \leqq 4\lambda_{\mathscr{D}}(z),$$

(11) $$|\nabla \log \lambda_{\mathscr{D}}(z)| \leqq 4/3d(z, \partial\mathscr{D})$$

for all $z \in \mathscr{D}$. Both inequalities are sharp.

Osgood commented that the extremal cases in Theorem C are quite different and neither inequality implies the other, this is true despite the sharp relations (4) between the Poincaré metric and the distance to the boundary. Moreover, there are some works concerning with the estimate of Poincaré metric (see, for example, [2] and [3]). Our second result shows that there is indeed some relationship between (10) and (11), it can be unified into an inequality, and (10), (11) are the special cases of it. We prove that

Theorem 2. *If $\mathscr{D} \subset \mathscr{C}$ is simply connected then*

(12) $$|\nabla \log \lambda_{\mathscr{D}}(z)| \leqq \frac{4}{d(z, \partial\mathscr{D})}(4\sqrt{d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z)} - 3d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z) - 1)$$

for all $z \in \mathscr{D}$ and the inequality is sharp.

Remark. Inequalities (10) and (11) in Theorem C also can be derived from Theorem 2.

In the case of multiply connected domain, Osgood proved that

Theorem D. *If $\mathscr{D} \subset \mathscr{C}$ is any domain then*

(13) $$|\nabla \log \lambda_{\mathscr{D}}(z)| \leqq 2/d(z, \partial\mathscr{D})$$

for all $z \in \mathscr{D}$.

Osgood remarked that it is an open question whether the constant 2 in (13) is sharp, but he showed that the constant 2 can not in general be replaced by $4/3$. In this case we prove that

Theorem 3. *If $\mathscr{D} \subset \mathscr{C}$ is any domain then there is a constant $a > 0$ such that*

(14) $$|\nabla \log \lambda_{\mathscr{D}}(z)| \leqq a/d(z, \partial\mathscr{D})$$

for all $z \in \mathscr{D}$ where $1.425 < a \leqq 2$.

2. The proofs of results. We need the following Theorem E, see [4] for the proof.

Theorem E. *Let $\mathscr{S}_d$, $1/4 \leqq d \leqq 1$, denote the class of functions $f(z) = z + a_2 z^2 + \dots$, regular and univalent in $\mathscr{B}$ where $d = \inf|\alpha|$, $f(z) \neq \alpha$ in the unit disk. Then*

$$a_2(d) = \max_{f \in \mathscr{S}_d}\{|a_2|\} = \frac{2}{d}(1 - \sqrt{d})(3\sqrt{d} - 1).$$

The proof of Theorem 1. Fix $z \in \mathscr{D}$ and choose a conformal mapping g of $\mathscr{B}$ onto $\mathscr{D}$ with $g(0) = z$. Then $f \circ g$ is a conformal mapping of $\mathscr{B}$ onto $f(\mathscr{D})$. Let $T_f(z) = f''(z)/f'(z)$, we have $T_{f\circ g}(w) = T_f(g(w))g'(w) + T_g(w)$ and

$$\frac{f''(z)}{f'(z)}g'(0)=\frac{(f\circ g)''(0)}{(f\circ g)'(0)}-\frac{g''(0)}{g'(0)}.$$

Hence,

$$\left|d(z,\partial\mathscr{D})\frac{f''(z)}{f'(z)}g'(0)\right|\leqq 4d(z,\partial\mathscr{D})+d(z,\partial D)\left|\frac{g''(0)}{g'(0)}\right|. \tag{15}$$

Let $F(w)=(g(w)-z)/g'(0)$, then $F(w)$ is a conformal mapping in $\mathscr{B}$, and $F'(0)-1=F(0)=0$, it is easily to see that $d_F(0,\partial F(\mathscr{B}))=d(z,\partial\mathscr{D})/|g'(0)|$, $F''(0)=g''(0)/g'(0)$, thus, by Theorem E, we have

$$\begin{aligned} d(z,\partial\mathscr{D})\,|g''(0)/g'(0)| &= d_F(0,\partial F(\mathscr{B}))\,|g'(0)F''(0)| \\ &\leqq 4\,|g'(0)|\,(1-\sqrt{d_F(0,\partial F(\mathscr{B}))})(3\sqrt{d_F(0,\partial F(B))}-1) \\ &= \frac{4}{\lambda_{\mathscr{D}}(z)}(4\sqrt{d(z,\partial\mathscr{D})\lambda_{\mathscr{D}}(z)}-3d(z,\partial\mathscr{D})\lambda_{\mathscr{D}}(z)-1), \end{aligned}$$

which combined with (15), implies (7).

The equality in (7) holds, because for every $d,\lambda>0$ with $1/4\leqq d\lambda\leqq 1$, there exist a domain $\mathscr{D}$, a function f and $z\in\mathscr{D}$ such that $\lambda_{\mathscr{D}}(z)=\lambda$ and $d(z,\partial\mathscr{D})=d$. To show this, let ϕ be a conformal mapping of $\mathscr{B}$ onto $\mathscr{D}$, then $|T_\phi(0)|\leqq 2\phi(\lambda(\phi(0))d(\phi(0),\partial\mathscr{D}))$ with $\Phi(t)=2(1-\sqrt{t})(3\sqrt{t}-1)/t$. Therefore, for every $d,\lambda>0$ with $1/4\leqq d\lambda\leqq 1$, there exist a domain $\mathscr{D}$ and a conformal mapping h satisfying $\lambda_{\mathscr{D}}(h(0))=\lambda$, $d(h(0),\partial\mathscr{D})=d$, and $T_h(0)=2\Phi(\lambda_{\mathscr{D}}(h(0))d(h(0),\partial\mathscr{D}))$. If we choose conformal mappings g of $\mathscr{B}$ onto $\mathscr{D}$ so that $T_g(0)=2\Phi(d\lambda)$, $\lambda_{\mathscr{D}}(g(0))=\lambda$ and $d(g(0),\partial\mathscr{D})=d$, and h of $\mathscr{B}$ onto $\mathscr{D}'$ so that $T_h(0)=-4$, then $f=h\circ g^{-1}$ which maps $\mathscr{D}$ onto $\mathscr{D}'$ satisfies $g'(0)T_f(g(0))=T_h(0)-T_g(0)=-4-2\Phi(d\lambda)$. So if we set $g(0)=z$, then we have $|T_f(z)|=\lambda_{\mathscr{D}}(z)(4+2\Phi(d\lambda))$, $\lambda_{\mathscr{D}}(z)=\lambda$ and $d(z,\partial\mathscr{D})=d$, which is (7) with equality. This completes the proof of Theorem 1.

Remark. It is easy to see that Theorem 1 is a generalization of Theorems A and B.

The proof of Theorem 2. Fix $z\in\mathscr{D}$ let $w=f(t)$ is the conformal mapping of $\mathscr{B}$ onto $\mathscr{D}$ such that $f(0)=z$. Then, from (9), we have $2\,|(\log\lambda_{\mathscr{D}}(f(t)))_w|=|f''(0)/f'(0)|\,|1/f'(0)|$. To estimate the value of $d(z,\partial\mathscr{D})\,|1/f'(0)|\,|f''(0)/f'(0)|$, we consider the function $F(t)=(f(t)-z)/f'(0)=t+a_2t^2+\dots$, let $\partial\mathscr{E}=F(\partial\mathscr{B})$ we get $d(z,\partial\mathscr{D})=d(0,\partial\mathscr{E})\,|f'(0)|$, and then $d(f(0),\partial\mathscr{D})\,|1/f'(0)|\,|f''(0)/f'(0)|=2d(0,\partial\mathscr{E})\,|a_2|$, by Theorem E, we obtain

$$\begin{aligned} d(z,\partial\mathscr{D})\,|1/f'(0)|\,|f''(0)/f'(0)| &\leqq 4(1-\sqrt{d(0,\partial\mathscr{E})})(3\sqrt{d(0,\partial\mathscr{E})}-1) \\ &= 4(1-\sqrt{d(z,\partial\mathscr{D})\lambda_{\mathscr{D}}(z)})(3\sqrt{d(z,\partial\mathscr{D})\lambda_{\mathscr{D}}(z)}-1) \\ &= 4(4\sqrt{d(z,\partial\mathscr{D})\lambda_{\mathscr{D}}(z)}-3d(z,\partial\mathscr{D})\lambda_{\mathscr{D}}(z)-1). \end{aligned}$$

The extremal function $f_d(z)$ given in Theorem E makes the inequality (12) becomes to equality for $\mathscr{D}=f_d(\mathscr{B})$ and $z=0$, thus (12) is sharp.

Remark. Let $y=d(z,\partial\mathscr{D})\lambda_{\mathscr{D}}(z)$, then $1/4\leqq y\leqq 1$, and put $F(y)=4\sqrt{y}-3y-1$, it is easy to find that $F(y)\leqq F(4/9)=1/3$. Therefore, we derive $|\nabla\log\lambda_{\mathscr{D}}(z)|\leqq 4/3d(z,\partial\mathscr{D})$, $z\in\mathscr{D}$. Moreover, putting (12) into the following

$$|\nabla\log\lambda_{\mathscr{D}}(z)|\leqq 4\lambda_{\mathscr{D}}(z)(\sqrt{1/d(z,\partial\mathscr{D})\lambda_{\mathscr{D}}(z)}-3-1/d(z,\partial\mathscr{D})\lambda_{\mathscr{D}}(z)),$$

and let $y=1/d(z,\partial\mathscr{D})\lambda_{\mathscr{D}}(z)$, then $1\leqq y\leqq 4$. For the function $F(y)=$

$4\sqrt{y} - 3 - y$, we have $F(y) \leqq F(4) = 1$, thus we again obtain $|\nabla \log \lambda_{\mathscr{D}}(z)| \leqq 4\lambda_{\mathscr{D}}(z)$, $z \in \mathscr{D}$. This gives that Theorem 2 generalizes Theorem C.

Now we will give an application of Theorem 2. If $\mathscr{D}$ is simply connected and γ is a hyperbolic geodesic in $\mathscr{D}$, let κ_e and κ_h denote the euclidean and hyperbolic curvatures of γ, the relation between κ_e and κ_h at any point $z_0 \in \gamma$ is that

$$\kappa_e = \lambda_{\mathscr{D}}\kappa_h + \partial \log \lambda_{\mathscr{D}}/\partial n \tag{16}$$

where $\partial/\partial n$ is the derivative in the normal direction at the point $z_0 \in \gamma$ (see [1], p. 455). Then by (12) and (16),

$$\begin{aligned}\kappa_e(z) = \partial \log \lambda_{\mathscr{D}}(z)/\partial n &\leqq |\nabla \log \lambda_{\mathscr{D}}(z)| \\ &\leqq \frac{4}{d(z, \partial\mathscr{D})}\left(4\sqrt{d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z)} - 3d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z) - 1\right)\end{aligned}$$

for $z \in \gamma$. At points where $\kappa_e(z) \neq 0$,

$$1/\kappa_e(z) \geqq d(z, \partial D)/4(4\sqrt{d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z)} - 3d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z) - 1) \geqq 3d(z, \partial\mathscr{D})/4,$$

thus, the euclidean circle of curvature to γ at z actually protrudes quite far over $\partial\mathscr{D}$, and it also has deep relation with $d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z)$. Hence (12) is also sharp form of Jørgensen's [5] result in the simply connected case.

The proof of Theorem 3. Let $\mathscr{D}$ be the image domain of $\mathscr{B}^x = \mathscr{B} - \{0\}$ under $f(z) = 1/2(z + 1/z)$, it is shown in [6] that $\lambda_{\mathscr{B}^x}(z) = 1/2|z| \log(1/|z|)$, and we calculate, from (9), that

$$\begin{aligned}((\log \lambda_{\mathscr{D}})_w \circ f(z)) f'(z) + f''(z)/2f'(z) &= (\log \lambda_{\mathscr{B}^x}(z))'_z \\ &= -1/2z + 1/2z \log(1/|z|).\end{aligned}$$

Thus we have

$$2((\log \lambda_{\mathscr{D}})_w \circ f(z)) = \frac{2}{f'(z)}\left(-\frac{f''(z)}{2f'(z)} - \frac{1}{2z} + \frac{1}{2z\log(1/|z|)}\right).$$

Let $z = ir$, $0 < r < 1$, we find that $d(f(ir), \partial\mathscr{D}) = (1/r - r)/2$ and

$$\begin{aligned}&d(f(ir), \partial\mathscr{D})\,|\nabla \log \lambda_{\mathscr{D}}(f(ir))| \\ &\qquad= \frac{2r(1-r^2)}{1+r^2}\left(\frac{1}{r(1+r^2)} - \frac{1}{2r} + \frac{1}{2r\log(1/r)}\right).\end{aligned}$$

Let $r = 1/2e$, we get $d(f(i/2e), \partial\mathscr{D})\,|\nabla \log \lambda_{\mathscr{D}}(f(i/2e))| = 1.4253363$. This shows that the constant a in Theorem 3 must be in $1.4253363 \leqq a \leqq 2$.

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