34. *Shimura Sums Related to Imaginary Quadratic Fields**)

By Ken ONO

Department of Mathematics, the University of Georgia, U. S. A.

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Abstract: Here we are interested in the arithmetic nature of sums of certain values of Hecke Grössencharakter, sums we call *Shimura sums*. In particular, we exhibit all primes p as the square root of a Shimura sum associated with a weight $k=3$ Hecke Grössencharakter of $K=\mathbf{Q}(\sqrt{-2})$. The formula shows that primes p come from ideals in O_K with norm $4p^2-n^2$. It will be shown that the non-existence of identities of this type imply certain cases of the Atkin Conjecture as well as Lehmer's Conjecture on the non-vanishing of Fourier coefficients.

Key words: modular forms; Shimura sums; complex multiplication.

1. Introduction. In an earlier paper [5], the author suggested that the arithmetic of ideals in imaginary quadratic number fields (*CM fields*) may be useful in studying the nature of the Lehmer and related conjectures on the nonvanishing of Fourier coefficients. This paper is the first realization of this idea.

First we give essential preliminaries and definitions. Let $K=\mathbf{Q}(\sqrt{-d})$ be an imaginary quadratic field with integer ring O_K with discriminant $-D$. A Hecke Grössencharakter ϕ of weight $k\geq 2$ with conductor Λ, an ideal in O_K, is defined in the following way. Let $I(\Lambda)$ denote the group of fractional ideals prime to Λ. We call a homomorphism ϕ

$$\phi : I(\Lambda)\to\mathbf{C}^*$$

satisfying

$$\phi(\alpha O_K)=\alpha^{k-1}\text{ when }\alpha\equiv 1 \bmod \Lambda$$

a Hecke Grössencharakter of weight k and conductor Λ. Define the power series $\Psi(\tau)$, in the variable $q=e^{2\pi i\tau}$, induced by a Hecke Grössencharakter ϕ by

$$\Psi(\tau)=\sum_{\mathfrak{a}}\phi(\mathfrak{a})q^{N(\mathfrak{a})}=\sum_{n=1}^{\infty}a(n)q^n,$$

where the sum is over ideals $\mathfrak{a}\subseteq O_K$ prime to Λ. Here $N(\mathfrak{a})$ is the ideal norm of $\mathfrak{a}$. The series $\Psi(\tau)$ belongs to a modular form of weight k on $\Gamma_0(D\,N(\Lambda))$ with Nebentypus character $\chi(n)=\left(\frac{-d}{n}\right)\frac{\phi(nO_K)}{n^{k-1}}$.

Using the notation defined above, it is clear that the Fourier coefficients $a(n)$ of $\Psi(\tau)$ are defined as sums of the Hecke Grössencharakter ϕ over the

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ideals in O_K with norm n that are relatively prime to the conductor Λ.

Note that if $p \in \mathbf{Z}$ is a prime which is inert (i.e. pO_K is a prime ideal), then $a(p) = 0$ since there are no ideals in O_K with norm p. Consequently, we know that the density of primes p with $a(p) = 0$ is one half. These modular forms have many interesting arithmetic properties; one may consult [2, 4, 6, 7] for more on CM forms.

Given a CM form $f(\tau)$, we will apply the Shimura lift to $f(4\tau)\Theta(\tau)$ where $\Theta(\tau)$ is the classical weight $\frac{1}{2}$ form

$$\Theta(\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}.$$

The version of the Shimura lift we use is due to Cipra and Selberg [1].

Let χ be a Dirichlet character $\bmod 4N$, and let t be a positive square-free integer. Let $F(\tau) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+\frac{1}{2}}(4N, \chi)$ where $k \in \mathbf{Z}^+$. Define $A_t(n)$ by the formal product:

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = L(s-k+1, \chi_t^{(k)}) \sum_{m=1}^{\infty} \frac{b(tm^2)}{m^s}. \tag{1}$$

Here $\chi_t^{(k)}(m) = \chi(m)\left(\frac{-1}{m}\right)^k\left(\frac{t}{m}\right)$ is a Dirichlet character $\bmod 4Nt$. Define $S_t(F)$, the image of $F(\tau)$ under the Shimura map S_t by

$$S_t(F) = \sum_{n=1}^{\infty} A_t(n)q^n.$$

The correspondence says that $S_t(F) \in S_{2k}(2N, \chi^2)$ if $k > 1$. If $k = 1$, then $S_t(F) \in M_{2k}(2N, \chi^2)$. The version of the Shimura lift we will use is contained in the following special case of a theorem due to Cipra.

Theorem 1. (*Cipra*) *Let $f(\tau) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(N, \chi)$ be an integral weight newform, and $F(\tau) \in S_{k+\frac{1}{2}}(4N, \chi_1^{(k)})$ defined by*

$$F(\tau) = \sum_{n=1}^{\infty} b(n)q^n = f(4\tau)\Theta(\tau).$$

Then the image of $F(\tau)$ by the Shimura map S_1 is

$$S_1(F) = f^2(\tau) - 2^{k-1}\chi(2)f^2(2\tau).$$

Furthermore, $S_1(F) \in S_{2k}(2N, \chi^2)$.

Using the notation from Theorem 1, we obtain the following formula for $A_1(n) = A(n)$ by (1):

$$A(n) = \sum_{d|n} d^{k-1}\chi_1^{(k)}(d)b\left(\frac{n^2}{d^2}\right). \tag{2}$$

Similarly, we obtain the following simple formula for $b(m^2)$:

$$b(m^2) = a\left(\frac{m^2}{4}\right) + 2\sum_{k=1}^{m-1} a\left(\frac{m^2-k^2}{4}\right). \tag{3}$$

Formula (3) motivates our definition of a Shimura sum.

Definition. Let $g(n)$ be an arithmetic function and c a positive integer. Then for $m \geq 1$ we define the *Shimura sum* $Sh(m, g, c)$ by:

$$Sh(m, g, c) = \sum_{k=1}^{m-1} g\left(\frac{m^2-k^2}{c}\right).$$

As an example of a Shimura sum, we mention Stieltjes' formula for $r_5(n)$, the number of unrestricted representations of a positive integer n as a sum of 5 squares [3]. If n is odd, then define $\sigma(n) = \sum_{d|n} d$. If n is even, then let $\sigma(n) = 0$. If p is an odd prime, then Stieltjes proved that

$$r_5(p) = 10\sigma(p^2) + 20Sh(p, \sigma, 1).$$

Now we return to the Shimura lift in terms of this new notation. In terms of Shimura sums, (2) becomes

$$A(n) = \sum_{d|n} d^{k-1}\chi_1^{(k)}(d) \left\{a\left(\frac{n^2}{4d^2}\right) + 2Sh\left(\frac{n}{d}, a, 4\right)\right\} \tag{4}$$

where a denotes the arithmetic function defined by the Fourier coefficients of $f(\tau)$.

2. The identities for $\mathbf{Q}(\sqrt{-2})$. In this section we present the identities using the general theory outlined in the previous section. The CM field we consider here is $K = \mathbf{Q}(\sqrt{-2})$. The relevant Hecke Grössencharakter ϕ has weight $k = 3$ with conductor $\Lambda = O_K$. By the theory presented above, the power series $\Psi(\tau)$

$$\Psi(\tau) = \sum_{\mathfrak{a}} \phi(\mathfrak{a}) q^{N(\mathfrak{a})} = \sum_{n=1}^{\infty} a(n) q^n$$

is a CM modular form with weight $k = 3$ on $\Gamma_0(8)$ with Nebentypus character $\chi(d) = \left(\frac{-2}{d}\right)$. This modular form has a convenient representation as product of Dedekind η-functions. Recall that $\eta(\tau)$ is a weight $k = \frac{1}{2}$ modular form with the following infinite product

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

It is easy to confirm that $\Psi(\tau) = \eta^2(8\tau)\eta(4\tau)\eta(2\tau)\eta^2(\tau)$ by a vector space dimension argument on $S_3(8, \chi)$.

As in Theorem 1, let $F(\tau) = \sum_{n=1}^{\infty} b(n) q^n$ be defined by

$$F(\tau) = \eta^2(32\tau)\eta(16\tau)\eta(8\tau)\eta^2(4\tau)\Theta(\tau),$$

the half integral weight form we are planning to lift. By Theorem 1, we find that

$$S_1(F) = \eta^4(8\tau)\eta^2(4\tau)\eta^2(2\tau)\eta^4(\tau),$$

the square of the original CM form. Here are the first few terms of the Fourier expansion of $S_1(F)$:

$$S_1(F) = \sum_{n=2}^{\infty} A(n) q^n = q^2 - 4q^3 + 16q^5 - 12q^6 - 8q^7 - 32q^9 + \dots$$

Upon further examination, it appears as though $A(4n) = 0$ for all integers n. Suppose that this conjecture is true, then (4) reduces to

$$0 = A(4n) = \sum_{d|n} d^2\chi(d) \left\{a\left(\frac{4n^2}{d^2}\right) + 2Sh\left(\frac{4n}{d}, a, 4\right)\right\} \tag{5}$$

because $\chi_1^{(2)} = \chi$ and $\chi(2m) = 0$ for all m.

In particular if $n = p$ is a prime, then from (5) we obtain

$$a(4p^2) + \chi(p)p^2 a(4) = -2Sh(4p, a, 4) - 2p^2\chi(p)Sh(4, a, 4).$$

It is easy to verify that $a(4) = 4$, and $a(16) = 16$. Therefore by the multiplicativity of the Fourier coefficients $a(n)$, we find

$$4a(p^2) + 4\chi(p)p^2 = -2Sh(4p, a, 4) - 2p^2\chi(p)Sh(4, a, 4).$$

Furthermore one easily verifies that $Sh(4, a, 4) = -2$; as a consequence we obtain the following identity for all primes p under the assumption that $A(4p) = 0$:

$$a(p^2) = -\frac{Sh(4p, a, 4)}{2}. \tag{6}$$

Since $\Psi(\tau) = \sum_{n=1}^{\infty} a(n)q^n$ is a newform, we can calculate the exact value of $a(p^2)$ once we know the value of $a(p)$, the eigenvalue of the $\Psi(\tau)$ with respect to the Hecke operator T_p. It is well known that newforms of weight k and Nebentypus χ satisfy

$$a(p^2) = a^2(p) - \chi(p)p^{k-1}. \tag{7}$$

Applying this identity to (6) produces

$$a^2(p) - \chi(p)p^2 = -\frac{Sh(4p, a, 4)}{2}. \tag{8}$$

At this point we state the main theorem which gives the desired identities.

Theorem 2. *With the notation above we have the following formulas.*

(i) *If p is inert in $K = \mathbf{Q}(\sqrt{-2})$, then*

$$p = \sqrt{-\frac{Sh(4p, a, 4)}{2}}.$$

(ii) *If p splits or ramifies in K, then*

$$p = \sqrt{a^2(p) + \frac{Sh(4p, a, 4)}{2}}.$$

Proof. From elementary number theory we recall the basic criterion for determining whether or not primes split in K. If p is inert in O_K then $\chi(p) = \left(\frac{-2}{p}\right) = -1$. Similarly if a prime p splits in O_K, then $\chi(p) = \left(\frac{-2}{p}\right) = 1$. Consequently if (8) is true, then (i) and (ii) follow easily. Recall that $a(p) = 0$ when p is inert because there are no ideals with norm p in K.

Therefore to prove the theorem it suffices to show that $A(4n) = 0$ for all $n \in \mathbf{Z}^+$. By Theorem 1, we know that the Fourier coefficients $A(n)$ are defined by

$$\eta^4(8\tau)\eta^2(4\tau)\eta^2(2\tau)\eta^4(\tau) = \sum_{n=2}^{\infty} A(n)q^n. \tag{9}$$

As an infinite product, (9) is

$$q^2 \prod_{n=1}^{\infty} (1 - q^{8n})^4(1 - q^{4n})^2(1 - q^{2n})^2(1 - q^n)^4 = \sum_{n=2}^{\infty} A(n)q^n. \tag{10}$$

We may disregard the factors whose exponents are always multiplies of 4; we only need to show that the coefficients $c(4n + 2) = 0$ where $c(n)$ is defined by:

$$\prod_{n=1}^{\infty} (1 - q^{2n})^2(1 - q^n)^4 = \sum_{n=2}^{\infty} c(n)q^n. \tag{11}$$

Fortunately these coefficients correspond very nicely with the Fourier expansion of the modular form $\eta^2(6\tau)\eta^4(3\tau) \in S_3(72, Id)$. Define two Hecke Grössencharakters ϕ_+ and ϕ_- by

$$\phi_+(\alpha O_K) = i^b\alpha^2 \text{ where } \alpha \equiv (1-i)^b \bmod (3O_K), \text{ and } b \in \mathbf{Z}/8\mathbf{Z}$$

and

$$\phi_-(\alpha O_K) = (-i)^b\alpha^2 \text{ where } \alpha \equiv (1-i)^b \bmod (3O_K), \text{ and } b \in \mathbf{Z}/8\mathbf{Z}.$$

Let $f_+(\tau)$ and $f_-(\tau)$ be weight $k = 3$ CM forms with respect to $\mathbf{Q}(i)$ induced by ϕ_+ and ϕ_- respectively. Here are the first few terms of these two forms:

$$f_+(\tau) = q + 2q^2 + 4q^4 - 8q^5 + 8q^8 - 16q^{10} - 10q^{13} + \dots$$

and

$$f_-(\tau) = q - 2q^2 + 4q^4 + 8q^5 - 8q^8 - 16q^{10} - 10q^{13} + \dots$$

The modular form $\eta^2(6\tau)\eta^4(3\tau)$ in terms of $f_+(\tau)$ and $f_-(\tau)$ is

$$\eta^2(6\tau)\eta^4(3\tau) = \sum_{n=1}^{\infty} d(n)q^n = -\frac{1}{2}\left(f_+\left(\tau + \frac{1}{2}\right) + f_-\left(\tau + \frac{1}{2}\right)\right). \tag{12}$$

Now we show that $f_+\left(\tau + \frac{1}{2}\right)$ and $f_-\left(\tau + \frac{1}{2}\right)$ have no terms with exponents of the form $n \equiv 3 \bmod 4$. If $p \equiv 3 \bmod 4$ is prime, then it is inert in $\mathbf{Q}(i)$. Therefore $d(p^r) = 0$ if r is odd by the recurrence (7). Moreover, if $n \equiv 3 \bmod 4$, then $d(n) = 0$ by the multiplicativity of Fourier coefficients of $f_+(\tau)$ and $f_-(\tau)$.

To prove the Theorem it suffices to show that $c(4n + 2) = 0$. However we know that $c(4n + 2) = d(12m + 7)$. Since $12n + 7 \equiv 3 \bmod 4$ the Theorem is proved.

3. Vanishing of Fourier Coefficients. By (4), it is clear that there are many formulas for the Fourier coefficients of Shimura lifts in terms of Shimura sums over Hecke Grössencharakters. In this vein we mention the conjectures on the nonvanishing of Fourier coefficients that are intimately related to the formulas like (4) coming from CM fields.

The most famous of these conjectures is Lehmer's conjecture on the nonvanising of $\tau(n)$, the Ramanujan function. Recall that $\tau(n)$ is defined by the infinite product

$$\sum_{n=1}^{\infty} \tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24}.$$

There are similar conjectures due to Atkin. Let $r > 0$ be an even integer other than 2, 4, 6, 8, 10, 14, and 26. Define $\tau_r(n)$ by

$$\sum_{n=0}^{\infty} \tau_r(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^r.$$

They conjecture that for these values of r that $\tau_r(n) \neq 0$ for all $n \in \mathbf{Z}^+$.

It is interesting to note that when $r = 12$, 16, and 24 we can apply our methods to get exact formulas for $\tau_r(n)$. Consequently the Atkin conjectures for $r = 12$, 16, and 24 say that there are no similar remarkable identities like the ones given in Theorem 2.

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