93. *A New Formula of Arc Length in Euclidean Space and its Application*

By Shigetake MATSUURA

RIMS, Kyoto University

(Communicated by Kiyosi ITÔ, M. J. A., Dec. 13, 1993)

Abstract: We shall introduce a new formula of the arc length of a rectifiable curve in the Euclidean space. By this new formula, the arc length is represented as a supremum of a linear functional on a subset of continuous functions defined on an interval of the real line, in which the parameter of the curve runs. This linearity enables us to calculate the arc length of admissible curves introduced in [1].

Key word: arc length.

1. New formula. Let $x=(x_1,\dots,x_n)$ be an orthonormal coordinate system of $\boldsymbol{R}^n$. Let Λ be a continuous curve parametrized by θ, $0 \leqq \theta \leqq 2\pi$ and given by the equations $x_j = x_j(\theta)$, $j=1,\dots,n$. By the points $0=\theta_0$, $\theta_1,\dots,\theta_N = 2\pi$ we divide up the interval $[0, 2\pi]$ into N sub-intervals $I_k = [\theta_{k-1}, \theta_k)$, $k=1,\dots,N$ of lengths $|I_k|$, $k=1,\dots,N$. We denote this division by Δ. Then the length $|\Lambda|$ of Λ is defined as

$$(1.1) \qquad |\Lambda| = \sup_{\Delta}\left\{\sum_{k=1}^{N}\Big(\sum_{j=1}^{n}(x_j(\theta_k)-x_j(\theta_{k-1}))^2\Big)^{1/2}\right\}.$$

If $|\Lambda|$ exists, the curve Λ is said to be *rectifiable*. It is well known that if Λ is rectifiable then $x_j(\theta)$, $j=1,\dots,n$ are continuous functions of bounded variation and therefore $\dot{x}_j$, $j=1,\dots,n$, derivatives of $x_j(\theta)$, $j=1,\dots,n$, in the sense of distributions of L. Schwartz are Radon measures of atom-free (having no point mass), that is, for every θ in $[0, 2\pi]$, $\dot{x}_j(\{\theta\})=0$, $j=1,\dots,n$. Hence $\dot{x}_j([\theta_{k-1},\theta_k]) = \dot{x}_j([\theta_{k-1},\theta_k))$, $j=1,\dots,n$, $k=1,\dots,N$. Thus we can regard $\dot{x}_j$, $j=1,\dots,n$ as continuous linear forms over $C([0,2\pi])$, the space of real-valued continuous functions on $[0, 2\pi]$. We denote these continuous forms by $\dot{x}_j[\phi]$, $j=1,\dots,n$ for ϕ in $C([0,2\pi])$ and these forms are also defined over the space of step functions. Then we have the following new formula of the arc length of a rectifiable curve:

Theorem 1.1.

$$(1.2) \qquad |\Lambda| = \sup_{\sum_{j=1}^n \phi_j^2 \leqq 1,\, \phi_j \in C([0,2\pi])} \Big(\sum_{j=1}^{n} \dot{x}_j[\phi_j]\Big) = \sup_{\sum_{j=1}^n \phi_j^2 \leqq 1,\, \phi_j \in C([0,2\pi])} \langle (\dot{x}_1,\dots,\dot{x}_n), (\phi_1,\dots,\phi_n)\rangle.$$

Remark. Obviously $C([0,2\pi])$ can be replaced by $C^\infty([0,2\pi])$.

Proof. Denote the right-hand side of (1.1) by A and the right-hand side of (1.2) by B.

First we shall show $A \leqq B$. Put

$$(1.3) \qquad \Sigma_\Delta = \sum_{k=1}^{N} (\sum_{j=1}^{n} (x_j(\theta_k) - x_j(\theta_{k-1}))^2)^{1/2}.$$

For the simplicity, we use the same notation I_k, $k = 1,\dots,N$ to denote both the intervals $[\theta_{k-1}, \theta_k)$ and their characteristic functions $Y(\theta_{k-1}) - Y(\theta_k)$, where $Y(\theta)$ is the Heaviside function. Since $\dot{x}_j$, $j = 1,\dots,n$ are Radon measures of atom-free, it implies that $\dot{x}_j[I_k] = x_j(\theta_k) - x_j(\theta_{k-1})$, $j = 1,\dots,n$, $k = 1,\dots,N$ and

$$(1.4) \qquad \Sigma_\Delta = \sum_{k=1}^{N} (\sum_{j=1}^{n} \dot{x}_j[I_k]^2)^{1/2}.$$

Since $(\sum_{j=1}^n \dot{x}_j[I_k]^2)^{1/2}$, $k = 1,\dots,N$ can be regarded as the lengths of vectors $(\dot{x}_1[I_k],\dots,\dot{x}_n[I_k])$, $k = 1,\dots,N$, there exist unit vectors $(a_1^k,\dots,a_n^k)$, $k = 1,\dots,N$ such that

$$\begin{aligned}(\sum_{j=1}^{n} \dot{x}_j[I_k]^2)^{1/2} &= \sum_{j=1}^{n} a_j^k \dot{x}_j[I_k] \\ &= \sum_{j=1}^{n} \dot{x}_j[a_j^k I_k],\ k = 1,\dots,N.\end{aligned}$$

Then

$$\Sigma_\Delta = \sum_{j=1}^{n} \dot{x}_j[\sum_{k=1}^{N} a_j^k I_k].$$

Put $\phi_j(\theta) = \sum_{k=1}^N a_j^k I_k(\theta)$, $j = 1,\dots,n$. Then

$$(1.5) \qquad \sum_{j=1}^{n} \phi_j(\theta)^2 \leqq 1 \text{ for } \theta \neq \theta_k,\ k = 1,\dots,N.$$

We approximate $\phi_j(\theta)$, $j = 1, \dots, n$ by continuous functions $\psi_j(\theta)$, $j = 1,\dots,n$ preserving

$$(1.6) \qquad \sum_{j=1}^{n} \psi_j(\theta)^2 \leq 1.$$

For example, for sufficiently small positive number η, we can take $\psi_{j,\eta}(\theta)$, $j = 1, \dots, n$ whose graphs on $[0, 2\pi]$ are defined by joining the following pairs of two points by straight line segments:

$(\theta_{k-1}, 0)$ and $(\theta_{k-1} + \eta, a_j^k)$, $k = 1,\dots,N$, $j = 1,\dots,n$;
$(\theta_{k-1} + \eta, a_j^k)$ and $(\theta_k - \eta, a_j^k)$, $k = 1,\dots,N$, $j = 1,\dots,n$;
$(\theta_k - \eta, a_j^k)$ and $(\theta_k, 0)$, $k = 1,\dots,N$, $j = 1,\dots,n$.

Since $\dot{x}_j$, $j = 1, \dots, n$ are atom-free, for every positive number ε, there exists a positive number η such that

$$|\sum_{j=1}^{n} \dot{x}_j[\phi_j] - \sum_{j=1}^{n} \dot{x}_j[\psi_{j,\eta}]| \leqq \varepsilon/2.$$

On the other hand, for every positive number ε, there exists Δ such that $A - \varepsilon/2 < \Sigma_\Delta$. Hence $A - \varepsilon \leqq \sum_{j=1}^n \dot{x}_j[\psi_{j,\eta}] \leqq B$. Since ε is arbitrary, it implies that $A \leqq B$.

Next we shall show $B \leqq A$. For every positive number ε, there exist continuous functions $\psi_j(\theta)$, $j = 1,\dots,n$ satisfying (1.6) such that $B - \varepsilon/2 < \sum_{j=1}^n \dot{x}_j[\psi_j]$. By the uniform continuity, $\psi_j(\theta)$, $j = 1,\dots,n$ can be approximated by step functions $\phi_j(\theta)$, $j = 1,\dots,n$ as follows. For these $\psi_j(\theta)$, $j = 1,\dots,n$ and every positive number η, there exists a division Δ: $0 = \theta_0 < \theta_1 < \cdots < \theta_N = 2\pi$ such that $|\psi_j(\theta_k) - \psi_j(\theta_{k-1})| \leqq \eta$, $j =$

$1,\dots,n$, $k=1,\dots,N$. By choosing suitable constants on the intervals $I_k = [\theta_{k-1}, \theta_k)$, $k=1,\dots,N$, we can find step functions $\phi_{j,\eta}(\theta) = \sum_{k=1}^{N} a_{j,\eta}^k I_k$, $j=1,\dots,n$ such that $|\phi_{j,\eta}| \leqq |\psi_j|$ and $|\psi_j(\theta) - \phi_{j,\eta}(\theta)| \leqq \eta$, $j=1,\dots,n$. If η is sufficiently small then

$$\left| \sum_{j=1}^{n} \dot{x}_j[\phi_{j,\eta}] - \sum_{j=1}^{n} \dot{x}_j[\psi_j] \right| \leqq \varepsilon/2.$$

Hence

$$\begin{aligned} B - \varepsilon &\leqq \sum_{j=1}^{n} \dot{x}_j[\phi_{j,\eta}] \\ &= \sum_{k=1}^{N}\sum_{j=1}^{n} a_{j,\eta}^k \dot{x}_j[I_k] \\ &= \sum_{k=1}^{N}\sum_{j=1}^{n} a_{j,\eta}^k (x_j(\theta_k) - x_j(\theta_{k-1})) \\ &\leqq \sum_{k=1}^{N} \left(\sum_{j=1}^{n} (a_{j,\eta}^k)^2\right)^{1/2} \left(\sum_{j=1}^{n} (x_j(\theta_k) - x_j(\theta_{k-1}))^2\right)^{1/2} \\ &\leqq A. \end{aligned}$$

This implies $B \leqq A$. Q.E.D.

2. Application to admissible curves. In [1], we introduced admissible curves, which are the generalization of curves of constant angle α in the plane. We shall state again here the definition and properties of admissible curves. Since admissible curves are closed, it is better to regard these curves are defined on the torus $\boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z}$ than on the interval $[0, 2\pi]$. For every fixed θ, let us consider the oriented straight line l_θ whose equation takes the canonical form:

(2.1) $$l_\theta : x\cos\theta + y\sin\theta = p(\theta)$$

and its orientation is defined as follows: Put $e_\theta = (\cos\theta, \sin\theta)$, $\tilde{e}_\theta = (-\sin\theta, \cos\theta)$. Then e_θ represents the direction perpendicular to the line l_θ and $\tilde{e}_\theta$ represents a direction parallel to l_θ. This pair $\{e_\theta, \tilde{e}_\theta\}$ determines the orientation of l_θ.

Definition 2.1. Let Λ be a closed continuous curve parametrized by $\theta \in \boldsymbol{T}$ and given by the equations $x = x(\theta)$ and $y = y(\theta)$, where $(x, y) \in \boldsymbol{R}^2$.

An oriented line l_{θ_0} having the canonical form:

(2.2) $$x\cos\theta_0 + y\sin\theta_0 = p(\theta_0)$$

is said to be a *generalized tangent line* to Λ through $(x(\theta_0), y(\theta_0))$ if the following two conditions are satisfied.

(2.3) $$x(\theta_0)\cos\theta_0 + y(\theta_0)\sin\theta_0 = p(\theta_0)$$

and

(2.4) $$\dot{x}(\theta)\cos\theta + \dot{y}(\theta)\sin\theta = 0 \text{ near } \theta = \theta_0$$

in the sense of distributions.

A closed continuous curve Λ is said to be an *admissible curve* if the following three conditions are satisfied.

(1) Λ is rectifiable.

(2) For every $\theta_0 \in \boldsymbol{T}$, the oriented straight line (2.2) is a generalized tangent line to Λ through $(x(\theta_0), y(\theta_0))$.

(3) $p(\theta) > 0$.

Here $p(\theta)$ is called a *generator of the admissible curve.* Since $p(\theta)$ is always to be supposed to be positive, $p(\theta)$ can be considered as the distance between the origin and the oriented line l_θ.

Remark. Obviously the condition (2) is equivalent to the following condition (2′):

(2′) For every $\theta \in \boldsymbol{T}$,

$$x(\theta)\cos\theta + y(\theta)\sin\theta = p(\theta) \tag{2.5}$$

and

$$\dot{x}(\theta)\cos\theta + \dot{y}(\theta)\sin\theta = 0 \text{ in } \mathscr{D}'(\boldsymbol{T}). \tag{2.6}$$

An admissible curve has the following property:

Theorem 2.2. *An admissible curve is a curve with C^1 generator $p(\theta)$ such that $p+\ddot{p}$ is a Radon measure.*

Proof. Since the left-hand side of (2.5) is continuous, $p(\theta)$ is also continuous in θ. Differentiate both sides of (2.5), then

$$\dot{p}(\theta) = -x(\theta)\sin\theta + y(\theta)\cos\theta + \dot{x}(\theta)\cos\theta + \dot{y}(\theta)\sin\theta.$$

Using (2.6) we have

$$\dot{p}(\theta) = -x(\theta)\sin\theta + y(\theta)\cos\theta. \tag{2.7}$$

This implies that $\dot{p}(\theta)$ is continuous, that is, $p(\theta)$ belongs to $C^1(\boldsymbol{T})$. Differentiate again both sides of (2.7), then

$$\ddot{p}(\theta) = -x(\theta)\cos\theta - y(\theta)\sin\theta - \dot{x}(\theta)\sin\theta + \dot{y}(\theta)\cos\theta.$$

Thus

$$p(\theta) + \ddot{p}(\theta) = -\dot{x}(\theta)\sin\theta + \dot{y}(\theta)\cos\theta. \tag{2.8}$$

This implies that $p+\ddot{p}$ is a Radon measure on $\boldsymbol{T}$. By (2.5) and (2.7), we have

$$\begin{pmatrix} p(\theta) \\ \dot{p}(\theta) \end{pmatrix} = T(-\theta)\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} \tag{2.9}$$

or

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = T(\theta)\begin{pmatrix} p(\theta) \\ \dot{p}(\theta) \end{pmatrix}, \tag{2.10}$$

where

$$T(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This is just the parametric representation of Λ by its generator $p(\theta)$. By (2.6) and (2.8), we have

$$\begin{pmatrix} 0 \\ p(\theta)+\ddot{p}(\theta) \end{pmatrix} = T(-\theta)\begin{pmatrix} \dot{x}(\theta) \\ \dot{y}(\theta) \end{pmatrix} \tag{2.11}$$

or

$$\begin{pmatrix} \dot{x}(\theta) \\ \dot{y}(\theta) \end{pmatrix} = T(\theta)\begin{pmatrix} 0 \\ p(\theta)+\ddot{p}(\theta) \end{pmatrix}. \tag{2.12}$$

Hence $\dot{x}$ and $\dot{y}$ are Radon measures if and only if $p+\ddot{p}$ is a Radon measure. Since Λ is rectifiable, it implies that $\dot{x}$ and $\dot{y}$ are Radon measures. Thus $p+\ddot{p}$ is a Radon measure. Q.E.D.

Denote by $M(\boldsymbol{T})$ the space of Radon measures, that is, the topological dual of $C(\boldsymbol{T})$. Applying Theorem 1.1, we have the following formula of the arc lengths of admissible curves:

Theorem 2.3. *Let Λ be an admissible curve with the generator $p(\theta)$. Denote by $|\Lambda|$ the arc length of Λ. Then*

(2.13) $$|\Lambda| = \|p + \ddot{p}\|_M.$$

Proof. Put

$$B = \sup_{\phi,\psi \in C^\infty, \phi^2+\psi^2 \leqq 1} (\dot{x}[\phi] + \dot{y}[\psi])$$

and

$$C = \|p + \ddot{p}\|_M = \sup_{\phi \in C^\infty, |\phi| \leqq 1} (p + \ddot{p})[\phi].$$

By Theorem 1.1, it is sufficient to show that $B = C$. First we shall show $C \leqq B$. For every C^∞ function ϕ satisfying $|\phi| \leqq 1$, (2.11) implies that

$$\begin{aligned}(p + \ddot{p})[\phi] &= (-\sin\theta \cdot \dot{x})[\phi] + (\cos\theta \cdot \dot{y})[\phi] \\ &= \dot{x}[-\sin\theta \cdot \phi(\theta)] + \dot{y}[\cos\theta \cdot \phi(\theta)] \leqq B,\end{aligned}$$

because $(-\sin\theta \cdot \phi(\theta))^2 + (\cos\theta \cdot \phi(\theta))^2 \leqq 1$. Hence $C \leqq B$.

Next we shall show $B \leqq C$. Fix C^∞ functions ϕ and ψ satisfying $\phi^2 + \psi^2 \leqq 1$ and denote $\Phi(\theta) = (\phi(\theta), \psi(\theta))$. Since $\{e_\theta, \tilde{e}_\theta\}$ is an orthonormal basis in $\boldsymbol{R}^2$, we have $\Phi(\theta) = \alpha(\theta)e_\theta + \beta(\theta)\tilde{e}_\theta$, where $\alpha(\theta) = \langle \Phi(\theta), e_\theta \rangle$, $\beta(\theta) = \langle \Phi(\theta), \tilde{e}_\theta \rangle$ and $\alpha^2 + \beta^2 \leqq 1$. Then (2.6) implies

$$\begin{aligned}&\dot{x}[\phi] + \dot{y}[\psi] \\ &= \dot{x}[\alpha(\theta)\cos\theta - \beta(\theta)\sin\theta] + \dot{y}[\alpha(\theta)\sin\theta + \beta(\theta)\cos\theta] \\ &= (\cos\theta \cdot \dot{x} + \sin\theta \cdot \dot{y})[\alpha] + (-\sin\theta \cdot \dot{x} + \cos\theta \cdot \dot{y})[\beta] \\ &= (p + \ddot{p})[\beta] \leqq \|p + \ddot{p}\|_M = C.\end{aligned}$$

Hence $B \leqq C$. Q.E.D.

Reference

[1] Matsuura, S.: On non-convex curves of constant angle. Functional Analysis and Related Topics, 1991. Lect. Notes in Math., vol. 1540, Springer-Verlag, pp. 251–268 (1993).