90. *Some Examples of Global Gevrey Hypoellipticity and Solvability*

By Todor GRAMCHEV*), Petar POPIVANOV*), and Masafumi YOSHINO**)

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1. **Notations and results.** Let $\boldsymbol{T}^2 := \boldsymbol{R}^2/\boldsymbol{Z}^2$ be the two dimensional torus, where $\boldsymbol{R}$ and $\boldsymbol{Z}$ are the sets of real numbers and integers respectively. We denote the variables in $\boldsymbol{T}^2$ by (x, y) and the differentiations on $\boldsymbol{T}^2$ by $\partial_x = \partial/\partial x$, and $\partial y = \partial/\partial y$. We denote by $C^\infty(\boldsymbol{T}^2)$ the set of smooth functions on $\boldsymbol{T}^2$. For $\sigma \geq 1$ we say that a function $f(x, y) \in C^\infty(\boldsymbol{T}^2)$ belongs to the Gevrey class $G^\sigma(\boldsymbol{T}^2)$ if for some $C > 0$

(1.1) $\quad |\partial_x^m \partial_y^n f(x, y)| \leq C^{m+n+1}(m!n!)^\sigma$, for all $m, n \in N$, $(x, y) \in \boldsymbol{T}^2$,

with the convention that $G^\infty(\boldsymbol{T}^2) := C^\infty(\boldsymbol{T}^2)$, if $\sigma = \infty$. We denote by $G^\sigma(\boldsymbol{T}^2)'$ the space of ultradistributions of class σ on $\boldsymbol{T}^2$. Clearly, $G^1(\boldsymbol{T}^2)$ is the set of analytic functions on $\boldsymbol{T}^2$ and $G^1(\boldsymbol{T}^2)'$ coincides with the class of periodic hyperfunctions on $\boldsymbol{T}^2$ (cf. [6] and [9]).

A differential operator P is said to be globally $G^\sigma(\boldsymbol{T}^2)$ solvable on $\boldsymbol{T}^2$ if for every $f \in G^\sigma(\boldsymbol{T}^2)$ there exists an ultradistribution $u \in G^\sigma(\boldsymbol{T}^2)'$ satisfying $Pu = f$. We say that P is globally $G^\sigma(\boldsymbol{T}^2)$ hypoelliptic if $u \in G^\sigma(\boldsymbol{T}^2)$ when $Pu \in G^\sigma(\boldsymbol{T}^2)$ and $u \in G^\sigma(\boldsymbol{T}^2)'$. The operator P is said to be locally G^σ solvable at a point $p \in \boldsymbol{T}^2$ if there exists a neighborhood U of p such that for every $f \in G_0^\sigma(U)$, there exists an ultradistribution $u \in G^\sigma(U)'$ such that $Pu = f$ in U. Similarly, we say that P is locally G^σ hypoelliptic at p if the following condition holds; if a point p does not belong to G^σ singular support of Pu then p does not belong to G^σ singular support of u.

In this note we shall give examples of first order operators with real coefficients on tori whose global properties are exotic in the following sense: Their global hypoellipticity and solvability in Gevrey class depend on Gevrey index σ. This makes a clear contrast to the known local results for operators of real principal type (cf. [5] and[1]). In fact, the first order analytic pseudodifferential operators of real principal type are not locally G^σ hypoelliptic for any $1 \leq \sigma \leq \infty$ and they are locally G^σ solvable for all $1 \leq \sigma \leq \infty$ (cf. [5] and [9]). In the global case, we have the following

Theorem 1 (Global hypoellipticity). *For every number σ, $1 \leq \sigma < \infty$ we can find infinitely many linearly independent real-valued functions $a \in G^1(\boldsymbol{T})$ such that the operators $P = \partial_x - a(x)\partial_y$ are globally $G^\theta(\boldsymbol{T}^2)$ hypoelliptic if $1 \leq \theta \leq \sigma$, while they are not globally $G^\theta(\boldsymbol{T}^2)$ hypoelliptic if $\sigma < \theta \leq \infty$.*

Theorem 2 (Global solvability). *For every number σ, $1 \leq \sigma < \infty$ we can*

*) Institute of Mathematics, Bulgarian Academy of Sciences, Bulgaria. Partially supported by contract MM-48/91 with MES of Bulgaria.

**) Faculty of Economics, Chuo University, Japan. Supported by JSPS and Chuo University Special Research Program.

find infinitely many linearly independent real-valued functions $a \in G^1(\boldsymbol{T})$ *such that the equations* $(\partial_x - a(x)\partial_y)u = f$, $f \in G^\theta(\boldsymbol{T}^2)$ *are always* $G^\theta(\boldsymbol{T}^2)$ *solvable for* f *such that* $\int_0^{2\pi}\int_0^{2\pi} f(x, y)dxdy = 0$ *if* $1 \leq \theta \leq \sigma$, *while they are not* $G^\theta(\boldsymbol{T}^2)$ *solvable for some* $f \in G^\theta(\boldsymbol{T}^2)$ *such that* $\int_0^{2\pi}\int_0^{2\pi} f(x, y)dxdy = 0$ *if* $\sigma < \theta \leq \infty$.

Remark. Theorems 1 and 2 are valid if we replace the inequalities $1 \leq \theta \leq \sigma$ and $\sigma < \theta \leq \infty$ by $1 \leq \theta < \sigma$ and $\sigma \leq \theta \leq \infty$, respectively. These facts can be proved by use of (ii) of Lemma which follows.

2. Proof of theorems. Theorems 1 and 2 are proved by constructing Liouville numbers with prescribed approximation rate by rational numbers. More precisely, we have

Lemma. *The following two properties are valid*:

(*i*) *For a given* $\sigma > 0$ *we can find an irrational number* t *such that for every* $0 < \varepsilon \ll 1$ *there exists* $C > 0$ *satisfying*

$$|p - tq| \geq C\exp(-\varepsilon q^{1/\sigma}) \text{ for any } p \in \boldsymbol{Z}, q \in N \tag{2.1}$$

while for any σ', $0 < \sigma < \sigma'$ *and any* $c > 0$ *there exist infinitely many* $p \in \boldsymbol{Z}$ *and* $q \in N$, p *and* q *relatively prime, such that*

$$|p - tq| \leq c\exp(-\varepsilon q^{1/\sigma'}). \tag{2.2}$$

(*ii*) *For a given* $\sigma > 0$ *we can find an irrational number* t *such that for every* $1 \leq \theta < \sigma$ *and every* $0 < \varepsilon \ll 1$ *there exists* $C > 0$ *satisfying*

$$|p - tq| \geq C\exp(-\varepsilon q^{1/\theta}) \text{ for any } p \in \boldsymbol{Z}, q \in N \tag{2.3}$$

while for any $c > 0$ *there exist infinitely many* $p \in \boldsymbol{Z}$ *and* $q \in N$, p *and* q *relatively prime, such that*

$$|p - tq| \leq c\exp(-\varepsilon q^{1/\sigma}). \tag{2.4}$$

All two types of numbers exhibited above, have the density of continuum.

Proof. We use the arguments of the paper of J. Leray and C. Pisot [8]. We shall give a sketch of the proof. We use the notations of [8]. First we observe that, if t exists we may assume $0 < t < 1$.

We shall define t by a continued fractions; $t = [a_1, a_2, \ldots, a_n, \ldots]$. Following (1.3) in [8] we introduce two sequences $\{p_n\}$ and $\{q_n\}$:

$$q_1 = 0,\ q_2 = 1,\ q_{n+2} = a_n q_{n+1} + q_n, \tag{2.5}$$

$$p_1 = 1,\ p_2 = 0,\ p_{n+2} = a_n p_{n+1} + p_n. \tag{2.6}$$

By (1.1) of [8], for every integer q such that $q_{n-1} \leq q \leq q_{n+1}$ we have

$$\inf_{p \in \boldsymbol{Z}} |p - tq| \geq \frac{1}{q_n} - \left| t - \frac{p_n}{q_n} \right| q, \tag{2.7}$$

where the equality is attained for $(q, p) = (q_{n-1}, p_{n-1})$ and (q_{n+1}, p_{n+1}). Therefore we have, for $q_{n-1} \leq q \leq q_{n+1}$

$$\inf_{p \in \boldsymbol{Z}} |p - tq| \geq \inf\{|p_{n-1} - tq_{n-1}|, |p_{n+1} - tq_{n+1}|\}, \tag{2.8}$$

where the equality is taken for $q = q_{n-1}$ and $q = q_{n+1}$. On the other hand we have

$$|p_{n+1} - tq_{n+1}| = \frac{1}{|\alpha_n q_{n+1} + q_n|}, \tag{2.9}$$

with α_n being defined by the relation (see (1.2) in [8]) $t = (\alpha_n p_{n+1} + p_n)/$

$(\alpha_n q_{n+1} + q_n)$. One checks easily that $\alpha_n = a_n + 1/\alpha_{n+1}$, $\alpha_n > 1$ (see (1.1) in [8]).

Let us assume that a_k, $0 \leq k \leq n-1$ are given. Then by (2.5) we define q_{n+1}. Next we choose and fix $a_n = [\exp(q_{n+1}^{1/\sigma}/(\ln q_{n+1}))]$, where $[r]$ stands for the integral part of $r \in \boldsymbol{R}$. On the other hand we recall (see (1.1) in [8]) that $\alpha_n = a_n + 1/\alpha_{n+1}$ and $\alpha_n > 1$. Then we easily see that for a given $0 < \delta \ll 1$ the quantity $\alpha_n q_{n+1} + q_n$ is estimated from below (respectively from above) by $(a_n - \delta)q_{n+1}$ (respectively by $(a_n + \delta)q_{n+1}$) when n is sufficiently large. Because of the consecutive construction of q_{n+2} and a_{n+1} we have that t is well defined and that there exist two positive constants C_1 and C_2 such that

$$C_1 q_{n+1} \exp\left(\frac{q_{n+1}^{1/\sigma}}{\ln q_{n+1}}\right) \leq |\alpha_n q_{n+1} + q_n| \leq C_2 q_{n+1} \exp\left(\frac{q_{n+1}^{1/\sigma}}{\ln q_{n+1}}\right),\ n \in N,$$

which proves part (i) of the lemma.

Concerning part (ii), we choose $a_n = [\exp(q_{n+1}^{1/\sigma})]$ for n sufficiently large. Then, by the last two-sided inequality we have the desired exponential growth.

The final statement for the density follows from the fact that all three estimates do not change when we replace a_n by $a_n + 1$ for infinitely many $n \in N$.

Sketch of the proof of Theorems. We note that $u(x, y) \in G^\sigma(\boldsymbol{T}^2)$ if and only if for some $c > 0$ and $C > 0$ the following estimate is true

$$|\partial_x^k \hat{u}(x, \eta)| \leq C^{k+1}(k!)^\sigma \exp(-c|\eta|^{1/\sigma}),\ k \in \boldsymbol{N},\ \eta \in \boldsymbol{Z},$$

where $\hat{u}(x, \eta)$ denotes the partial Fourier transform of u with respect to y. By the partial Fourier transform with respect to y the equation $Pu := (\partial_x - a(x)\partial_y)u = f$ is equivalent to $\hat{P}\hat{u} = (\partial_x - ia(x)\eta)\hat{u} = \hat{f}$. We set $2\pi\tau_a = \int_0^{2\pi} a(x)dx$, $\Lambda(x) := \int_0^x a(t)dt$. We assume that τ_a is positive and irrational. Then the periodic solution to the equation $\hat{P}\hat{u} = \hat{f}$ is given by

(2.10)
$$\hat{u}(x, \eta) = e^{i\eta\Lambda(x)}\left(\frac{e^{2\pi i\eta\tau_a}}{1 - e^{2\pi i\eta\tau_a}}\int_0^{2\pi} e^{-i\eta\Lambda(t)}\hat{f}(t, \eta)dt + \int_0^x e^{-i\eta\Lambda(t)}\hat{f}(t, \eta)dt\right),$$

for $\eta \neq 0$. If $a(x)$ is real-valued this expression implies that P is globally hypoelliptic and solvable in G^σ for f such that $\int_0^{2\pi}\int_0^{2\pi} f(x, y)dx\,dy = 0$ if and only if for every $0 < \varepsilon \ll 1$ there exists $C > 0$ such that

$$\left|\tau_a - \frac{p}{q}\right| \geq C\exp(-\varepsilon q^{1/\sigma}),\ p \in \boldsymbol{Z},\ q \in N. \tag{2.11}$$

Indeed, (2.11) follows from the estimate of the denominator $1 - e^{2\pi i\eta\tau_a}$ in (2.10).

Hence our theorem is proved if we choose c to be a number t satisfying the statement (i) of Lemma and we choose $a(x)$ such that $\int_0^{2\pi} a(x)dx = 2\pi c$. This proves Theorems.

Remark. Let t be a transcendental number constructed in the proof of

(i) of Lemma with $\sigma = 1$. Then the equation $Pu := (\partial_x - t\partial_y)u = f$ is solvable for $f \in G^1(\boldsymbol{T}^2)$ such that $\int_0^{2\pi}\int_0^{2\pi} f(x, y)\,dxdy = 0$. On the other hand, for every $\sigma > 1$ it is not solvable for some $f \in G^\sigma(\boldsymbol{T}^2)$ such that $\int_0^{2\pi}\int_0^{2\pi} f(x, y)\,dxdy = 0$. We remark that in view of the definition of periodic hyperfunctions the solution exists in the class of periodic hyperfunctions even in the case $\sigma > 1$ (cf. (2.11) and Proposition 2.4.4 of [6]).

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