86. *Recurrence of a Diffusion Process in a Multidimensional Brownian Environment**)

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Introduction. Let $\boldsymbol{W}$ be the space of continuous functions on $\boldsymbol{R}^d$ vanishing at the origin. In this paper an element of $\boldsymbol{W}$ is called an environment. Given an environment W, we consider a diffusion process $\boldsymbol{X}_W = \{X(t),\ t \geq 0,\ P_W^x,\ x \in \boldsymbol{R}^d\}$ with generator

$$\frac{1}{2}(\Delta - \nabla W \cdot \nabla) = \frac{1}{2} e^W \sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(e^{-W} \frac{\partial}{\partial x_k}\right).$$

When W is bounded, the result of Nash [8] for fundamental solutions of parabolic equations guarantees the existence of a diffusion process $\boldsymbol{X}_W^0$ with generator

$$\sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(e^{-W} \frac{\partial}{\partial x_k}\right).$$

For a general W we still have a nice diffusion process $\boldsymbol{X}_W^0$ (e.g. see [4]) and hence $\boldsymbol{X}_W$ can be constructed from $\boldsymbol{X}_W^0$ through a random time change. Without any assumption on the behavior of $W(x)$ for large $|x|$ the process $\boldsymbol{X}_W$ may explode within a finite time, but such a case is excluded automatically since we are interested in the recurrence of $\boldsymbol{X}_W$. We consider the probability measure $\boldsymbol{P}$ on $\boldsymbol{W}$ with respect to which $\{W(x),\ x \in \boldsymbol{R}^d,\ P\}$ is a Lévy's Brownian motion with a d-dimensional time. The collection of diffusion processes $\boldsymbol{X} = \{\boldsymbol{X}_W\}$ in which W is allowed to vary as a random element in $(\boldsymbol{W}, P)$ is called a diffusion in a d-dimensional Brownian environment. When $d = 1$ this was considered by Brox [1] and Schumacher [9] as a diffusion model exhibiting the same asymptotic behavior as Sinai's random walk in a random environment ([10]); see also [11] for some refined results. Recently Mathieu [7] obtained some very interesting results concerning a long time asymptotic problem for $\boldsymbol{X}$ in the case $d \geq 2$. Motivated by [7] the present paper was written.

In this paper we prove that $\boldsymbol{X}_W$ is recurrent for almost all Brownian environments W in any dimension d, namely, for any nonnegative Borel function f on $\boldsymbol{R}^d$ such that $f > 0$ on a set of positive Lebesgue measure the equality

$$P_W^x\left\{\int_0^\infty f(X(t))\,dt = \infty\right\} = 1,\ x \in \boldsymbol{R}^d,$$

holds for almost all W with respect to $\boldsymbol{P}$. In [3] Fukushima, Nakao and Takeda discussed the same problem but with the replacement of $W(x)$ by $\tilde{W}(|x|)$,

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where $\tilde{W}(t)$ is a Brownian motion with a 1-dimensional time.

To obtain our result we employ Ichihara's recurrence criterion ([4]) which, in the present special case, asserts that X_W^0 (W is fixed) is recurrent if

$$\int_1^\infty \left\{\int_{S^{d-1}} e^{-W(r\theta)} d\theta\right\}^{-1} r^{-d+1} dr = \infty,$$

where $d\theta$ is the uniform distribution on S^{d-1}. We can also employ Fukushima's recurrence criterion ([2]) which, in the present special case, asserts that X_W (W is fixed) is recurrent if there exists a sequence $\{u_n\}$ such that $0 \le u_n \le 1$, $\lim u_n = 1$ a.e. and $\lim \mathscr{E}(u_n, u_n) = 0$, where $\mathscr{E}(u, v)$ is the Dirichlet form associated with X_W, namely,

$$\mathscr{E}(u, v) = \frac{1}{2}\int_{\boldsymbol{R}^d} \nabla u \cdot \nabla u\, e^{-W} dx.$$

Since it is obvious that X_W is recurrent if and only if X_W^0 is recurrent, either criterion yields our result. But for the verification of these criteria we need some information on the asymptotic behavior of $W(x)$ for large $|x|$. A key point in obtaining this informaiton is to consider the one-parameter family $\{T_t, t \in \boldsymbol{R}\}$ of measure preserving transformations on $(\boldsymbol{W}, P)$ defined by (1. 3) and then to use its ergodicity.

§1. Brownian motion with a d-dimensional time. Let $d \ge 2$ and as before let P be the probability measure on $\boldsymbol{W}$ such that $\{W(x), x \in \boldsymbol{R}^d, P\}$ is a Brownian motion with a d-dimensional time ([6: p. 277]), that is, a Gaussian system with

(1.1) $$E\{W(x)\} = 0,\ W(0) = 0,$$

(1.2) $$E\{W(x)W(y)\} = \frac{1}{2}\{|x| + |y| - |x - y|\}.$$

For each $t \in \boldsymbol{R}$ and $W \in \boldsymbol{W}$ we define an element $T_t W$ of $\boldsymbol{W}$ by

(1.3) $$(T_t W)(x) = e^{-t/2} W(e^t x), \quad x \in \boldsymbol{R}^d.$$

Then $\{T_t, t \in \boldsymbol{R}\}$ is a one-parameter family of measure preserving transformations on the probalility space $(\boldsymbol{W}, P)$. Using (1. 2) we can easily compute the covariance matrix of

$$e^{-t/2}W(e^t x_1),\ e^{-t/2}W(e^t x_2), \cdots, e^{-t/2}W(e^t x_m),\ W(x_1'),\ W(x_2'), \cdots, W(x_n')$$

for fixed $t \in \boldsymbol{R}$ and $x_1, \ldots, x_m, x_1', \ldots, x_n' \in \boldsymbol{R}^d$, and the following lemma can be proved in the same way as in Itô [5].

Lemma 1. *$\{T_t, t \in \boldsymbol{R}\}$ is mixing and hence ergodic.*

Next let $0 < a < b$, put $K = \{x \in \boldsymbol{R}^d : a \le |x| \le b\}$ and consider the Banach space $B = C(K)$, the space of real valued continuous functions on K, and the real Hilbert space $H = L^2(K, dx)$. The inner product in H is denoted by $\langle \cdot, \cdot \rangle$. Regarding $W_K = \{W(x), x \in K\}$ as an H-valued random variable, we denote by γ the probability distribution of W_K. Since every Borel set in the space B is also a Borel set in the space H and since W_K is regarded as a B-valued random variable, we have $\gamma(B) = 1$. γ is a Gaussian measure on H with

(1.4a) $$\int_H e^{\langle f, g\rangle} \gamma(dg) = E\left\{\exp \int_K f(x) W(x) dx\right\} = \exp\left\{\frac{1}{2}\langle Af, f\rangle\right\}, f \in H,$$

(1.4b) $$Af(x)=\int_K \frac{1}{2}\{|x|+|y|-|x-y|\}f(y)dy.$$

For $\phi=Af_0$ with $f_0\in H$ we define the ϕ-transform γ_ϕ by $\gamma_\phi(\Gamma)=\gamma(\{g:g+\phi\in\Gamma\})$. Then the following Cameron-Martin formula is easily verified by using (1. 4).

(1.5) $$\gamma_\phi(dg)=\exp\left\{\langle f_0, g\rangle-\frac{1}{2}\langle Af_0, f_0\rangle\right\}\gamma(dg).$$

Lemma 2. *Any nonempty open set in the space B has a positive* γ-measure.

Proof. We first prove that the range $R=\{Af:f\in H\}$ is dense in B. If this were not true, there exists a finite signed measure $\mu\neq 0$ on K such that

(1.6) $$\int_K Af(x)\mu(dx)=0 \quad \text{for all} \quad f\in H.$$

Since the left hand side of (1. 6) equals $\langle f, g\rangle$ where

$$g(x)=\int_K \frac{1}{2}\{|x|+|y|-|x-y|\}\mu(dy)\in H,$$

(1.6) implies $g=0$. Therefore, regarding μ as a signed measure in $\boldsymbol{R}^d$ we have

(1.7) $$\int_{\boldsymbol{R}^d}\int_{\boldsymbol{R}^d}\frac{1}{2}\{|x|+|y|-|x-y|\}\mu(dy)\mu(dy)=0.$$

In the same way as in the proof of Théorème 58 of [6: p. 276] we can prove that the left hand side of (1. 7) equals

$$\text{const.}\int_{\boldsymbol{R}^d}|\xi|^{-d-1}|\hat{\mu}(\xi)-\hat{\mu}(0)|^2 d\xi,$$

where $\hat{\mu}(\xi)$ is the Fourier transform of μ. Therefore μ must be concentrated on $\{0\}$. But this is impossible because $0\notin K$ and hence R must be dense in B. Next we notice that the whole space B, which has γ-measure 1, can be expresses as a union of a countable number of open balls of the form $B_\varepsilon(\phi)=\{\psi\in B:\|\psi-\phi\|_\infty<\varepsilon\}$, $\phi\in R$, $\varepsilon>0$ being arbitrary but fixed. On the other hand by the Cameron-Martin formula (1.5) $\gamma(B_\varepsilon(\phi))=\gamma_{-\phi}(B_\varepsilon(0))>0$ if and only if $\gamma(B_\varepsilon(0))>0$ provided that $\phi\in R$. Therefore we must have $\gamma(B_\varepsilon(\phi))>0$ for all $\phi\in R$. This implies the assertion of the lemma.

§2. Recurrence of X_W. Since our result in the 1-dimensional case is easily obtained from a general theory of 1-dimensional diffusion processes, we assume $d\geq 2$.

Theorem 1. *X_W is recurrent for almost all Brownian environments W.*

Proof. It is enough to prove that X_W^0 is recurrent for almost all Brownian environments W and, according to Ichihara's criterion ([4: Theorem A]) it is also enough to prove that

(2.1) $$\int_1^\infty\left\{\int_{S^{d-1}}e^{-W(r\theta)}d\theta\right\}^{-1}r^{-d+1}dr=\infty,\ P\text{-}a.s.$$

If we put $M(t)=\min\{(T_tW)(\theta):\theta\in S^{d-1}\}$, then

(2.2) the left hand side of (2. 1)

$$= \int_0^\infty e^{(2-d)t} \left\{ \int_{S^{d-1}} \exp(-e^{t/2}(T_t W)(\theta)) d\theta \right\}^{-1} dt$$

$$\geq \int_0^\infty e^{(2-d)t} \exp\{e^{t/2} M(t)\} dt \geq \int_0^\infty \mathbf{1}_{(a,\infty)}(M(t)) dt,$$

provided that $a > 0$ is chosen so that $(2-d)t + ae^{t/2} \geq 0$ holds for all $t \geq 0$. Next take $K = \{x \in \boldsymbol{R}^d : 1 \leq |x| \leq 2\}$ and consider B, H and γ as in the preceeding section. Since $\Gamma = \{\phi \in B : \min(\phi(x) : |x| = 1) > a\}$ is an open set in B, we have $\gamma(\Gamma) > 0$ by Lemma 2. The ergodicity of $\{T_t, t \in \boldsymbol{R}\}$ now implies

$$\lim_{T \to \infty} T^{-1} \int_0^T \mathbf{1}_{(a,\infty)}(M(t)) dt = E\{\mathbf{1}_{(a,\infty)}(M(0))\} = \gamma(\Gamma) > 0, \ P\text{-}a,s.,$$

and hence $\int_0^\infty \mathbf{1}_{(a,\infty)}(M(t)) dt = \infty$, P-$a.s.$, which combined with (2.2) proves (2. 1).

Remark 1. $\boldsymbol{X}_W$ is null-recurrent (P-$a.s.$) in the sense that $m_W(dx) = e^{-W} dx$ is an invariant measure for $\boldsymbol{X}_W$ with $m_W(\boldsymbol{R}^d) = \infty$.

Remark 2. Fukushima's criterion can also be used for proving Theorem 1; in fact, by virtue of Lemmas 1, 2 it is still easy to prove the existence of a sequence of radial functions u_n in $C_0^\infty(\boldsymbol{R}^d)$ such that $0 \leq u_n \leq 1$, $\lim u_n = 1$ a.e. and $\mathcal{E}(u_n, u_n) = 0$. This argument also proves the recurrence of $\boldsymbol{X}_{|W|}$ for almost all Brownian environments W.

Remark 3. $\boldsymbol{X}_{-|W|}$ is recurrent for $d = 1$ and transient for $d \geq 2$ for almost all Brownian environments W. The proof in the case $d \geq 2$ is as follows. According to Theorem B of [4] the transience of $\boldsymbol{X}^0_{-|W|}$ (and consequently of $\boldsymbol{X}_{-|W|}$) follows if one proves that, for almost all Brownian environments W,

$$\int_0^\infty e^{-|W(r\theta)|} r^{-d+1} dr < \infty \tag{2.3}$$

for θ belonging to some subset (which may depend on W) of S^{d-1} with a positive uniform measure. But this can be proved by showing that the expectation (with respect to P) of the left hand side of (2. 3) is finite for each fixed θ.

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