78. *Gauss Decomposition of Connection Matrices and Application to Yang-Baxter Equation. II*

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We follow the same terminologies as in [1].

1. Gauss decomposition of G. Case where $m = 2$. The matrix $G = G(x \mid \alpha_1)$ depends on x_2/x_1 and of size $n+1$. We denote by $g_{n-i,n-j} = g_{n-i,n-j}(x_2/x_1)$ its entries as

(1.1) $$g_{n-i,n-j} = (Y^+_{i,n-i} : \mathrm{reg}\ Y^-_{j,n-j})_{\Phi^{(a)}_{n,2}}$$

where the corresponding summits $\xi = v^+_{i,n-i}$ and $\eta = v^-_{j,n-j}$ are given by $\xi_k = x_1 q^{1+(k-1)\gamma}$ $(1 \le k \le i)$, $x_2 q^{1+(k-i-1)\gamma}$ $(1+i \le k \le n)$ and $\eta_k = x_1 q^{-\beta-(k-1)\gamma}$ $(1 \le k \le j)$, $x_2 q^{-\beta-(k-i-1)\gamma}$ $(1+j \le k \le n)$ respectively.

First we present a few basic properties of the principal connection matrix G.

Lemma 1.

(1.2) $$\tau_1 G(x \mid \alpha_1) = {}^tG(x \mid \alpha_1) = S'^{-1}_{\tau_1} \cdot G(x \mid \alpha_1) \cdot S'_{\tau_1}$$

where ${}^tG(x \mid \alpha_1)$ denotes the transposed matrix and S'_{τ_1} denotes the matrix with only non-zero $(i, n-i)$th components $a_{i,n-i}\left(\frac{x_2}{x_1}\right)$,

$$a_{i,n-i}(u) = u^{-2\gamma i(n-i)} q^{\gamma^2 i(n-i)(-n+2i)+i(n-i)} \frac{\theta(q^{-i\gamma}u)_{\hat{\imath}}\,\theta(q^{(1-i)\gamma}u)_{\hat{\imath}}}{\theta(q^{1-(n-i)\gamma}u^{-1})_{\hat{\imath}}\,\theta(q^{1-(n-i-1)\gamma}u^{-1})_{\hat{\imath}}}$$

for $\hat{\imath} = \min(i, n-i)$. In particular $a_{0,n}(u) = a_{n,0}(u) = 1$.

(1.3) $$S'_{\tau_1} = \Lambda^{-1} S_{\tau_1} \tau_1 \Lambda, \text{ for } \Lambda = \mathrm{Diag}[\lambda_0, \ldots, \lambda_n]$$

where $\lambda_i = \lambda_i(x_2/x_1) = \theta(q^{1-i\gamma}x_2/x_1)_{\hat{\imath}}\,\theta(q^{1-(i-1)\gamma}x_2/x_1)_{\hat{\imath}} \left(\frac{x_2}{x_1}\right)^{\hat{\imath}-\gamma i(n-i)} q^{i^2(n-i)\gamma^2 - i\hat{\imath}\gamma}$

and S_{τ_1} denotes the matrix with only non zero $(i, n-i)$th components 1 *so that* $S^2_{\tau_1} = 1$.

(1.4) $$G(x \mid \alpha_1)^{-1} = (q^{2n(\beta^2+\beta)}/(1-q)^{2n})\, M \cdot G(x^{-1} \mid -\alpha_1 - 2\beta + 2(n-1)(\gamma-1)) \cdot M'$$

where M and M' denote the diagonal matrices $M = \mathrm{Diag}[\mu_0, \ldots, \mu_n]$, $M' = \mathrm{Diag}[\mu'_0, \ldots, \mu'_n]$ *such that* $\mu_{n-i} = \left(\frac{x_2}{x_1}\right)^{2i\beta} a_i\left(\frac{x_2}{x_1}\right) a_{n-i}\left(\frac{x_1}{x_2}\right) a_{n-i,i}\left(\frac{x_2}{x_1}\right)$, $\mu'_{n-i} = \left(\frac{x_2}{x_1}\right)^{-2i\beta} a_i\left(\frac{x_1}{x_2}\right) a_{n-i}\left(\frac{x_2}{x_1}\right) a_{n-i,i}\left(\frac{x_1}{x_2}\right)$. *Here $a_i(u)$ denotes*

(1.5) $$a_i(u) = q^{i(i-1)\beta\gamma + \gamma^2(i-1)i(i+1)/3 + \gamma(i-1)i/2} \cdot \frac{\theta(q^{1+\gamma})_i\,\theta(q^{1+\beta})_i\,\theta(q^{1+\beta}u)_i}{\theta'(1)^i\,\theta(q^{1+\gamma})^i\,\theta(qu)_i},$$

where $\theta(u)_i$ denotes the product $\theta(u)\theta(uq^{\gamma}) \cdots \theta(uq^{(i-1)\gamma})$ *and* $\theta'(1) =$

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$\left[\frac{d\theta(u)}{du}\right]_{u=1} = -(q)_\infty^3$. *Remark that* $a_{i,n-i}(u) \cdot a_{n-i,i}(u^{-1}) = 1$.

We want to define the special cycles $\mathrm{reg}\, Y_i^- \otimes Y_{n-i}^+ (0 \leq i \leq n)$ which play a central role in our discussion.

Definition 1. The cycle $Y_i^- \otimes Y_{n-i}^+$ is defined as the set of $t \in (\boldsymbol{C}^*)^n$ such that $t_j = x_1 q^{-\beta-\nu_j-(j-1)\gamma} (1 \leq j \leq i)$, $= x_2 q^{1+\nu_j+(j-i-1)\gamma} (i+1 \leq j \leq n)$ for $\nu_j \in \boldsymbol{Z}_{\geq 0}$. Its regularization $\mathrm{reg}\, Y_i^- \otimes Y_{n-i}^+$ is constructed such that the Jackson integral of $\Phi_{n,2}^{(a)}(t)$ over the cycle $\mathrm{reg}\, Y_i^- \otimes Y_{n-i}^+$ can be obtained by taking residues of $\Phi_{n,2}^{(a)}(t)$ along the loci $t_1 = x_1 q^{-\beta-\nu_1}, \ldots, t_i = x_1 q^{-\beta-\nu_i-(i-1)\gamma}$ (see for details [2]).

The cycles $\mathrm{reg}\, Y_i^- \otimes Y_{n-i}^+$ are characterized by the following asymptotics.

Lemma 2. *For* $|x_1| \gg |x_2|$, *we have*

$$(1.6) \quad \int_{\mathrm{reg}\, Y_i^- \otimes Y_{n-i}^+} \Phi_{n,2}^{(a)}(t)\tilde{\omega} \sim \Pi_{k=1}^i \left(\frac{x_2}{x_1} q^{\beta+(k-1)\gamma}\right)^\beta \frac{\theta(q^{-\beta-(k-1)\gamma} x_1/x_2)}{\theta(q^{-(k-1)\gamma} x_1/x_2)}$$

$$\cdot B_i^{(-)}(\alpha_1 + \beta + n - i)\, B_{n-i}^{(+)}(\alpha_1 + i(1-2\gamma))\, x_1^{\alpha_1+\cdots+\alpha_i+i\beta+(n-i)i}\, x_2^{\alpha_{i+1}+\cdots+\alpha_n}.$$

The meaning of Gauss decomposition can be explained as follows. In the homology $H_n((\boldsymbol{C}^*)^n, \Phi_{n,2}^{(a)}, \partial_q)$ associated with Jackson integrals (2.4) in [1], we have two kinds of relations,

$$(1.7) \quad Y_{i,n-i}^+ = \sum_{j=0}^n g_{n-i,n-j}\, \mathrm{reg}\, Y_{j,n-j}^-$$

$$(1.8) \quad \mathrm{reg}\, Y_i^- \otimes Y_{n-i}^+ = \sum_{i\leq j\leq n} \omega_{n-i,n-j}^{(n)}\, \mathrm{reg}\, Y_{j,n-j}^-$$

$$(1.9) \quad = \sum_{0\leq j\leq i} \omega_{n-i,n-j}^{*(n)}\, Y_{n-i,n-j}^+.$$

These identities give the Gauss decomposition stated in [1]. Concerning explicit description of $\omega_{n-i,n-j}^{(n)} = \omega_{n-i,n-j}^{(n)}\left(\frac{x_2}{x_1} \mid \alpha_1\right)$ and $\omega_{n-i,n-j}^{*(n)} = \omega_{n-i,n-j}^{*(n)}\left(\frac{x_2}{x_1} \mid \alpha_1\right)$ in terms of theta rational functions we can state the following theorem.

Theorem 1. *The elements* $\omega_{n-i,n-j}^{(n)}$ *and* $\omega_{n-i,n-j}^{*(n)}$ *are expressed in terms of theta monomials as follows.*

$$(1.10)\, \omega_{n,n-i}^{(n)}\left(\frac{x_2}{x_1} \mid \alpha_1\right) = g_{n,n-i}(x \mid \alpha_1) = (1-q)^n q^{-\frac{1}{2}n(n-1)\gamma - i(n-i)\gamma(\alpha_1-(n+i-1)\gamma)+C_n}$$

$$\cdot \left(\frac{x_2}{x_1}\right)^{i(\alpha_1-(i-1)\gamma)} \frac{(q)_\infty^{3n}\, \theta(q^{\alpha_1+2+\beta-(n-1)\gamma} x_2/x_1)_i \theta(q^{1+\gamma})^i}{\theta(q^{1+\alpha_1-2(n-1)\gamma})_n \theta(q^{2+\beta} x_2/x_1)_i \theta(q^{1+\gamma})_i}$$

$$\cdot \frac{\theta(q^{\alpha_1+2+\beta-(n-1)\gamma})_{n-i}\, \theta(q^{1+\gamma})^{n-i}\, \theta(q^{1+(1-i)\gamma} x_1/x_2)_{\hat{i}}}{\theta(q^{2+\beta})_{n-i} \theta(q^{1+\gamma})_{n-i}\, \theta(q^{-(n-i)\gamma} x_2/x_1)_{\hat{i}}}$$

for $C_n = -\frac{2}{3} n(n-1)(2n-1)\gamma^2 - \frac{n(n-1)}{2}\gamma + n\beta(\alpha_1 - (n-1)\gamma) + n\alpha_1(1 + (n-1)\gamma)$.

$$(1.11) \quad \omega_{n-i,n-j}^{(n)}\left(\frac{x_2}{x_1} \mid \alpha_1\right) = \omega_{n-i,n-j}^{(n-i)}(q^{i\gamma} x_2/x_1 \mid \alpha_1 + i(1-2\gamma))$$

$$(1.12) \quad \omega_{n-i,n-j}^{*(n)}\left(\frac{x_2}{x_1} \mid \alpha_1\right) = (1-q)^{-2i} \left(\frac{x_2}{x_1}\right)^{2(i-j)\beta} q^{2i(\beta^2+\beta)}$$

$$\cdot a_i\left(\frac{x_2}{x_1}\right) a_{n-i}^{-1}\left(\frac{x_2}{x_1}\right) a_j\left(\frac{x_1}{x_2}\right) a_{n-j}\left(\frac{x_2}{x_1}\right) \omega_{i,j}^{(n)}\left(\frac{x_2}{x_1} \mid \tilde{\alpha}_1\right),$$

for $\tilde{\alpha}_1 = -\alpha_1 - 2\beta + 2(n-1)(\gamma - 1)$. *In particular we have* $\omega^{(n)}_{0,0} = \omega^{*(n)}_{n,n} = 1$.

Indeed (1.10) is an immediate consequence of [1](3.4) and (3.5). (1.11) is obtained from (1.10) by reducing it to lower dimensional Jackson integrals with respect to the variables $t_{i+1}, \dots, t_n$, while $t_1 = q^{-\beta-\nu_1}x_1, \dots, t_i = q^{-\beta-\nu_i-(i-1)\gamma}x_1$ are fixed. (1.12) is obtained from (1.4) and (1.8) by the substitution $t_1 \to t_1^{-1}, \dots, t_n \to t_n^{-1}$.

Hence the one cocycle $\{W_e, W_{\tau_1}\}$ defined by [1](1.3) has the expression, $W_e = 1$ and

(1.13) $$W_{\tau_1} = \Omega \cdot S'_{\tau_1} \cdot (\tau_1 \Omega)^{-1} = \Omega \Lambda^{-1} \cdot S_{\tau_1} \cdot (\tau_1(\Omega\Lambda^{-1}))^{-1}$$

so that we have $W_{\tau_1} \cdot \tau_1 W_{\tau_1} = 1$.

2. Yang-Baxter equation. Case where $m \geq 3$. As beforehand, G is of size $\binom{m+n-1}{m-1} \times \binom{m+n-1}{m-1}$ and has the similar properties to the ones in Lemma 1.

(2.1) $$\tau G(x \mid \alpha_1) = S'^{-1}_{\tau}\, G(x \mid \alpha_1) S'_{\tau_1}$$

where the one cocycle $\{S'_\tau\}_{\tau \in \mathfrak{S}_m}$ corresponds to a symmetric representation $\{S_\tau\}_{\tau \in \mathfrak{S}_m}$ of $\mathfrak{S}_m$ induced by permutations among $\alpha-$ unstable cycles reg $Y^-_{f_1,\dots,f_m}$,

(2.2) $$S'_\tau = \Lambda(x)^{-1}\, S_\tau (\tau\Lambda(x)^{-1})^{-1}$$

for a suitable diagonal matrix $\Lambda(x)$. We can introduce the lexicographic ordering into the set of partitions $F = \langle f_1, \dots, f_m \rangle$ as follows.

We say that $\langle f_1, \dots, f_m \rangle$ is greater than $\langle f'_1, \dots, f'_m \rangle$ if and only if there exists an integer r such that $f_r > f'_r$, $f_{r+1} = f'_{r+1}, \dots, f_m = f'_m$. By using this ordering we can define the lower and upper triangular matrices Ω and Ω^* respectively such that the corresponding entries $\omega_{F,F'}$ and $\omega^*_{F,F'} = 0$ according as $F < F'$ and $F > F'$ respectively. The cycles $Z_F = \sum_{F \geq F'} \omega^{(n)}_{F,F'}$ reg $Y^-_{F'} = \sum_{F \leq F'} \omega^{*(n)}_{F,F'} Y^+_{F'}$ give characteristic asymptotics of corresponding Jackson integrals [1](2.4) for $|x_1| \gg \cdots \gg |x_m|$ (see also [5] in relation to quantum KZ equations). G has the Gauss decomposition and

(2.3) $$W^{(m)}_\tau = \Omega\, S'_\tau\, (\tau\Omega)^{-1} = (\Omega\Lambda^{-1})\, S_\tau (\tau(\Omega\Lambda^{-1}))^{-1}$$

defines a one-cocycle. Under this circumstance,

Theorem 2. *For two partitions* $F = \langle f_1, \dots, f_m \rangle$ *and* $F' = \langle f'_1, \dots, f'_m \rangle$ *of* n *we have the expression for the* (F, F') *element* $w_{r;F,F'}$ *of the matrix* $W_{\tau_r} = ((w_{r;F,F'}))_{F,F'}$ *as*

(2.4) $$w_{r;F,F'} = \delta_{f_1 f'_1} \cdots \delta_{f_{r-1} f'_{r-1}} \delta_{f_{r+2} f'_{r+2}} \cdots \delta_{f_m f'_m}$$

$$w^{(2)}_{r;f_{r+1},f'r+1}\left(\frac{x_{r+1}}{x_r} \mid \alpha_1 + (f_1 + \cdots + f_{r-1})(1-2\gamma) + (m-r-1)\beta + (n - f_1 - \cdots - f_{r+1})\right).$$

The matrix $W^{(2)}_r = ((w^{(2)}_{r;i,j}))^{f_r+f_{r+1}}_{i,j=0}$ is of order $f_r + f_{r+1} + 1$, where $f_1, \dots, f_{r-1}, f_{r+2}, \dots, f_m$ and $f'_1, \dots, f'_{r-1}, f'_{r+2}, \dots, f'_m$ being fixed such that $f_1 = f'_1, \dots, f_{r-1} = f'_{r-1} f_{r+2} = f'_{r+2}, \dots, f_m = f'_m$. This is the connection matrix for Jackson integrals of the function $\Phi^{(a)}_{r,r+1}$ depending only on x_r, x_{r+1}, (2.5) $\Phi^{(a)}_{r,r+1}(t_k\,; f_1+\cdots+f_{r-1}+1\leq k\leq f_1+\cdots+f_{r+1})$

$$=\Pi_{f_1+\cdots+f_{r-1}+1\leq k\leq f_1+\cdots+f_{r+1}} t_k^{a_k+(m-r-1)\beta+n-f_1-\cdots-f_{r+1}} \frac{(t_k/x_r)_\infty (t_k/x_{r+1})_\infty}{(t_k q^\beta/x_r)_\infty (t_k q^\beta/x_{r+1})_\infty}$$

$$\cdot\,\Pi_{f_1+\cdots+f_{r-1}+1\leq i<j\leq f_1+\cdots+f_{r+1}} \frac{(q^{\gamma'}t_j/t_i)_\infty}{(q^{\gamma}t_j/t_i)_\infty}(t_i-t_j).$$

Hence the matrix $W^{(2)}_r(x)=W^{(2)}_r(x_r,x_{r+1})$ can be written as in (1.13), where n and $t_1,\ldots,t_n$ should be replaced by f_r+f_{r+1}, $t_{f_1+\cdots+f_{r-1}+1},\ldots,t_{f_1+\cdots+f_{r+1}}$ respectively. In this way we get *the one cocycle condition for* $\{W^{(m)}_{\tau_r}\}_{1\leq r\leq m-1}$ *which coincides with Yang-Baxter equation* in view of (1.2) and (1.4) in [1].

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