76. *Resonance in the Cauchy Problem of a Parabolic Equation*

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1. Introduction. Let q be a natural number, and we consider the Cauchy problem of the following strongly parabolic equation of $2q$-th order:

(1) $$\frac{\partial u}{\partial t} = ((-1)^{q-1} + b(t, x)) \frac{\partial^{2q} u}{\partial x^{2q}} \quad t > 0, \quad x \in \mathcal{R}^1,$$

(2) $$u(0, x) = u_0(x), \quad x \in \mathcal{R}^1,$$

where an initial data u_0 and a coefficient b satisfy Assumption 1 (this and the other terminology are defined at §2). In [6], it is proved;

Proposition. *Let Assumption* 1 *hold, then there exists a unique wide sense solution* u *of* (1) *with* (2). *In addition there is a constant* c_∞ *such that*

(3) $$\lim_{t \to \infty} \| u(t, \cdot) - c_\infty \|_0 = 0.$$

Thus in the present note, we announce that c_∞ can be calculated from b and u_0, and its value changes drastically whether u_0 *resonates* with b or not.

On c_∞, only a few results have been known. If one of the following (a) and (b) hold:

(a) $q = 1$, b is real valued and independent of t, and for a constant $\overline{u_0}$,
$$u_0 - \overline{u_0} \in \mathcal{L}_1(\mathcal{R}^1),$$

(b) b is independent of x and there is a constant $\overline{u_0}$ such that
$$\overline{u_0} = \lim_{L \to +\infty} \frac{1}{L} \int_0^L u_0(x)\, dx = \lim_{L \to +\infty} \frac{1}{L} \int_{-L0}^0 u_0(x)\, dx,$$

then it is known in [3, 4, etc.] and [1] that

(4) $$c_\infty = \overline{u_0}.$$

But (4) does not make clear delicate relation between c_∞ and b, because the both conditions above prevent that u_0 *resonates* with b. In this sense, (4) is very different from our result.

Our method to calculate c_∞ is based on an extended Girsanov type formula. The usual Girsanov formula is well known in the theory of probability. It works when first order terms are added to a second order parabolic equation. Besides it, we introduced the extended Girsanov type formula in [5], which works when same order terms are added to a $2q$-th order parabolic equation. By this formula, the wide sense solution u of (1) with (2) is represented in a series, which enables us to calculate c_∞.

2. Notations. Let $\lambda \geq 0$, and let $\mathcal{M}^\lambda(\mathcal{R}^1)$ be a set of all complex valued measures $\mu(d\xi)$ such that

$$\| \mu \|_\lambda \equiv \int_{\mathcal{R}^1} (1 + |\xi|)^\lambda |\mu|(d\xi) < \infty$$

where $|\mu|$ denotes total variation of μ. As well known, $\mathcal{M}^\lambda(\mathcal{R}^1)$ is a Banach algebra under convolution $*$ and norm $\|\mu\|_\lambda$.

We denote by $\mathscr{F}^\lambda(\mathscr{R}^1)$ a Banach space of all Fourier transforms of $\mathscr{M}^\lambda(\mathscr{R}^1)$ i.e. $f \in \mathscr{F}^\lambda(\mathscr{R}^1)$ is written as (5) for a $\mu_f \in \mathscr{M}^\lambda(\mathscr{R}^1)$, and we define $\|f\|_\lambda \equiv \|\mu_f\|_\lambda$. Note that $\mathscr{F}^0(\mathscr{R}^1)$ contains the Schwartz class, constants, etc.

Put $\mathscr{R}^+ = [0, \infty)$ and let $\mathscr{M}^0(\mathscr{R}^+, \mathscr{R}^1)$ be a set of all complex valued measures $\eta(t, d\xi)$, $t \in \mathscr{R}^+$ such that

(a) $\eta(t, \xi) \in \mathscr{M}^0(\mathscr{R}^1)$ for each $t \in \mathscr{R}^+$

(b) $\|\eta(t,\cdot) - \eta(s,\cdot)\|_0 \to 0$ as $t \to s$ on $\mathscr{R}^+$.

As before, $\mathscr{F}^0(\mathscr{R}^+, \mathscr{R}^1)$ denotes a set of all Fourier transforms of $\mathscr{M}^0(\mathscr{R}^+, \mathscr{R}^1)$, that is functions which are written as (6) for an $\eta \in \mathscr{M}^0(\mathscr{R}^+, \mathscr{R}^1)$.

Throughout the note, we suppose:

Assumption 1. (a) $u_0 \in \mathscr{F}^0(\mathscr{R}^1)$, *that is*

$$u_0(x) = \int \exp\{i\xi x\}\, \mu_0(d\xi) \text{ for a } \mu_0 \in \mathscr{M}^0(\mathscr{R}^1). \tag{5}$$

(b) $b \in \mathscr{F}^0(\mathscr{R}^+, \mathscr{R}^1)$, *that is*

$$b(t, x) = \int \exp\{i\xi x\}\, \eta_b(t, d\xi) \text{ for an } \eta_b \in \mathscr{M}^0(\mathscr{R}^+, \mathscr{R}^1). \tag{6}$$

(c) *In* (6), η_b *has a structure*

$$\eta_b(t, d\xi) = h_b(t, \xi)\, \nu_b(d\xi), \tag{7}$$

where a continuous function $h_b(t, x)$ *and* $\nu_b(d\xi) \in \mathscr{M}^0(\mathscr{R}^1)$ *satisfy*

$$1 \geq \sup_{(t,\xi)\in\mathscr{R}^+\times\mathscr{R}^1} |h_b| \quad \text{and} \quad 1 > \|\nu_b\|_0. \tag{8}$$

Next we specify a solution of the Cauchy problem of (1).

Definition 2. A function $v(t, x) \in \mathscr{F}^0(\mathscr{R}^+, \mathscr{R}^1)$ is called *a wide sense solution* of (1) with (2), if there exists a sequence

$$\{(v^{(m)}(t, x), u_0^{(m)}(x))\,;\, m \geq 1\} \subset \mathscr{F}^0(\mathscr{R}^+, \mathscr{R}^1) \times \mathscr{F}^{2q}(\mathscr{R}^1)$$

such that;

(a) $\lim_{m\to\infty} \|u_0^{(m)} - u_0\|_0 = 0$ and

$\lim_{m\to\infty} \sup_{0<t<T} \|v^{(m)}(t,\cdot) - v(t,\cdot)\|_0 = 0$ for any $T > 0$.

(b) For each $\partial^{2q}v^{(m)}/\partial x^{2q}$, $\partial v^{(m)}/\partial t \in \mathscr{F}^0(\mathscr{R}^+, \mathscr{R}^1)$, and $v^{(m)}$ is a classical solution of (1) with an initial condition $u(0, x) = u_0^{(m)}(x)$ instead of (2).

3. A combination of resonance. For the measures in (5) and (7), we define

$$K(u_0) \equiv \{y \in \mathscr{R}^1\,;\, |\mu_0|(\{y\}) > 0\} - \{0\} \tag{9}$$

$$K(b) \equiv \{z \in \mathscr{R}^1\,;\, |\nu_b|(\{z\}) > 0\} - \{0\}. \tag{10}$$

Note that $K(u_0)$ and $K(b)$ are both countable sets at most, by Assumption 1.

Definition 3. Take natural numbers m_k, $k = 1, \ldots, l$, a point $y \in K(u_0)$, and points z_k's $\in K(b)$, $k = 1, \ldots, l$, such as $z_1 < z_2 < \cdots < z_l$. If it holds that

$$y + m_1 z_1 + m_2 z_2 + \cdots + m_l z_l = 0,$$

then an ordered set

$$\tilde{\gamma} \equiv (y\,;\, \underbrace{z_1, \ldots, z_1}_{m_1}, \underbrace{z_2, \ldots, z_2}_{m_2}, \ldots, \underbrace{z_l, \ldots, z_l}_{m_l})$$

is called *a combination of resonance*. We denote by Γ a whole of all combinations of resonance, and say that u_0 *resonates with* b if $\Gamma \neq 0$.

Theorem 4. *Let Assumption* 1 *hold. If* u_0 *does not resonate with* b, *i.e.* $\Gamma = \emptyset$, *then*

$$c_\infty = \mu_0(\{0\}).$$

Remark 5. (a) If $K(u_0) = \emptyset$, then u_0 does not resonate with any b. For $K(u_0) = \emptyset$, it is sufficient that

$$u_0(x) = \int \exp\{i\zeta x\}\, \widehat{u_0}(\zeta)\, d\zeta \text{ for a } \widehat{u_0}(\zeta) \in \mathscr{L}_1(\mathscr{R}^1).$$

(b) If $K(b) = \emptyset$, then any u_0 is not resonate with b. It is sufficient for $K(b) = \emptyset$ that

$$b(t, x) = \int \exp\{i\xi x\}\, h_b(t, \xi)\, \hat{b}(\xi)\, d\xi \text{ for a } \hat{b}(\xi) \in \mathscr{L}_1(\mathscr{R}^1).$$

Example 1. For a natural number n, consider

$$\frac{\partial u}{\partial t} = \left((-1)^{q-1} + \frac{1}{2}\sin x\right)\frac{\partial^{2q} u}{\partial x^{2q}}, \quad t > 0,\ x \in \mathscr{R}^1,$$

$$u(0, x) = u_0(x) \equiv \sin\left(1 + \frac{1}{n+1}\right)x, \quad x \in \mathscr{R}^1.$$

Here $K(u_0) = \left\{1 + \frac{1}{n+1},\ -1 - \frac{1}{n+1}\right\}$, and $K(b) = \{1, -1\}$. So u_0 does not resonate with b, and $c_\infty = 0$ even if n is very large. Compare this with Example 2 in §4.

§4. Resonance. Consider an ordered set $\mathscr{C} \equiv (x_0; x_1, x_2, \cdots, x_j)$ consisting of points in $\mathscr{R}^1$. For $\mathscr{C}$, we define a number $Q(\mathscr{C})$ as follows:

Definition 6. Case 1. If one of the following numbers

(11) $$x_0,\ x_0 + x_1,\ x_0 + x_1 + x_2, \ldots,\ x_0 + x_1 + \cdots + x_{j-1}$$

is zero, then we define $Q(\mathscr{C}) = 0$.

Case 2. If none of (11) is zero, then we define

$$Q(\mathscr{C}) = \mu_0(\{x_0\})\, \nu_b(\{x_1\}) \cdots \nu_b(\{x_j\}) \times$$
$$\times \lim_{T\to\infty} \frac{1}{T} \int \cdots \int_{0<s_1<\cdots<s_j<t<T} ds_1 \cdots ds_j\, dt\, h_b(s_1, x_1) \cdots h_b(s_j, x_j)$$
$$\times (ix_0)^{2q} \exp\{-x_0^{2q} s_1\}$$
$$\times (i(x_0 + x_1))^{2q} \exp\{-(x_0 + x_1)^{2q}(s_2 - s_1)\}$$
$$\times \cdots \times (i(x_0 + \cdots + x_{j-1}))^{2q} \exp\{-(x_0 + \cdots + x_{j-1})^{2q}(s_j - s_{j-1}\}\}.$$

Remark 7. (a) $Q(\mathscr{C})$ exists for any $\mathscr{C}$, by Assumption 1.

(b) If the coefficient b does not depend on t, that is $h_b \equiv 1$, then the above integrations can be carried out, and we get

$$Q(\mathscr{C}) = (-1)^{qj} \mu_0(\{x_0\})\, \nu_b(\{x_1\}) \cdots \nu_b(\{x_j\}).$$

Now we are in a position to state our remained assertion.

Theorem 8. *Let Assumption* 1 *hold. If* u_0 *resonates with* b, *then*

(12) $$c_\infty = \mu_0(\{0\}) + \sum_{\tilde{\gamma}\in\Gamma} \tilde{\sum}_{\tilde{\gamma}} Q(\tilde{\gamma}),$$

where $\tilde{\sum}_{\tilde{\gamma}}$ *denotes to take summation over all permutations of a combination of resonance* $\tilde{\gamma}$ *except its first element* y, *that is all permutations of*

$$(\underbrace{z_1, \ldots, z_1}_{m_1},\ \underbrace{z_2, \ldots, z_2}_{m_2}, \ldots, \underbrace{z_l, \ldots, z_l}_{m_l}).$$

Remark 9. (a) The right-hand side of (12) always converges by Assumption 1.

(b) Compare the following Examples 2 and 3 with Example 1 in §3, and we see that c_∞ is very sensible with respect to a little change of u_0.

(c) All argument in the note can be extended to multidimensional cases.

Example 2. We consider

$$\frac{\partial u}{\partial t} = \left((-1)^{q-1} + \frac{1}{2}\sin x\right)\frac{\partial^{2q} u}{\partial x^{2q}}, \qquad t > 0, \quad x \in \mathscr{R}^1, \tag{13}$$

$$u(0, x) = u_0(x) \equiv \sin x, \qquad x \in \mathscr{R}^1. \tag{14}$$

Now $K(u_0) = \{1, -1\} = K(b)$, and there are infinite combinations of resonance's. So following to Definition 6 and Theorem 8, we get

$$| c_\infty - (-1)^q \times 0.2675\cdots | \leq 0.014\cdots.$$

Here it should be noted that if $q = 1$, we happen to calculate c_∞ for (13) and (14) by the well known ergodic property of a diffusion process on a circle. So we get

$$c_\infty = \sqrt{3} - 2 = -0.2679\cdots.$$

Example 3. Again we treat (13) with

$$u(0, x) = u_0(x) \equiv \cos x, \quad x \in \mathscr{R}^1$$

instead of (14). $K(u_0)$ and $K(b)$ are same as in Example 2, and u_0 resonates with b, but (12) derives that

$$c_\infty = 0.$$

Example 4. Let us treat a second order equation of a time depending coefficient:

$$\frac{\partial u}{\partial t} = \left(1 + \frac{1}{2}\sin t \sin x\right)\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad x \in \mathscr{R}^1,$$

$$u(0, x) = u_0(x) \equiv \sin x, \qquad x \in \mathscr{R}^1.$$

$K(u_0)$ and $K(b)$ are same as in Example 2, and we get

$$| c_\infty + 0.1178\cdots | \leq 0.0104\cdots.$$

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