71. On the Intersection of Continuous Local Tents

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Let X be an n-dimensional Euclidean space with inner product $(-\mid-)$ and norm $|-|$. The open unit ball $\{x \in X ; |x| < 1\}$ is denoted by B. For a convex set $M \subset X$ we denote by aff M the smallest affine subspace of X containing M. Then there exists a unique vector subspace lin M of X such that aff $M = x + \operatorname{lin} M$ for any $x \in$ aff M. The interior of M with respect to the subspace aff M is called the *relative interior* of M and denoted by rint M.

Next, let $K_1, \ldots, K_s$ be convex cones with common vertex at x_0. The system $K_1, \ldots, K_s$ is said to be *separable* if one of these cones can be separated by a hyperplane from the intersection of the others.

Further, let $\Omega \subset X$ and $x_0 \in \Omega$. A convex cone $Q \subset X$ with vertex at x_0 is called a *tent* (*continuous tent, smooth tent*) at x_0 of Ω if there exists a mapping (continuous mapping, smooth mapping) $\psi : X \to X$ defined in a neighborhood of x_0 such that

1) $\psi(x) = x + o(x - x_0)$, where $\lim_{\xi \to 0} \frac{o(\xi)}{|\xi|} = 0$ and $o(0) = 0$,

2) for some $\varepsilon > 0$, $\psi(Q \cap (x_0 + \varepsilon B)) \subset \Omega$.

More generally, a convex cone $K \subset X$ with vertex at x_0 is called a *local tent* (*continuous local tent, smooth local tent*) if, for any $\bar{x} \in K$, there exists a tent (continuous tent, smooth tent) $Q \subset K$ at x_0 of Ω such that $\bar{x} \in$ rint Q and aff $Q =$ aff K.

The notion of continuous and smooth tents (or local tents) is effectively used in deriving necessary conditions for optimality in various optimization problems. For obtaining these necessary conditions, fundamental roles are played by the following

Theorem A. *Let $\Omega_1, \ldots, \Omega_s$ be subsets of X having a point x_0 in common and $K_1, \ldots, K_s$ an inseparable system of continuous local tents at x_0 of $\Omega_1, \ldots, \Omega_s$, respectively. Assume that $K_j \neq$ off K_j for some j. Then there exists a point $x_1 \in \Omega := \cap_{j=1}^{s} \Omega_j$ other than x_0.*

By a subtle algebraic topological method, V. G. Boltjanskiǐ ([1]) gave a proof of Theorem A, printed in small type on a total of 14 pages.

Theorem A, however, is a simple consequence of the following

Theorem B (theorem on the intersection of continuous local tents). *Let $\Omega_1, \ldots, \Omega_s$ be subsets of X having a point x_0 in common and $K_1, \ldots, K_s$ an inseparable system of continuous local tents at x_0 of $\Omega_1, \ldots, \Omega_s$, respectively. Then $K := \cap_{j=1}^{s} K_j$ is a* (*not necessarily continuous*) *local tent at x_0 of $\Omega := \cap_{j=1}^{s} \Omega_j$.*

In fact, we may assume $x_0 = 0$. If $K_{j0} \neq \operatorname{lin} K_{j0}$ and $0 \in$ rint K_{j0}, then there exists a $\delta > 0$ such that $\delta B \cap \operatorname{lin} K_{j0} \subset K_{j0}$. But this implies $K_{j0} = \operatorname{lin} K_{j0}$. Therefore $0 \notin \operatorname{rint} K_{j0}$. As $\operatorname{rint} K = \cap_{j=1}^{s} \operatorname{rint} K_j \neq \phi$ by the inseparability, so there exists a $\xi_0 \in K$ such that $\xi_0 \neq 0$. By Theorem B, the convex cone K is a local tent at 0 of Ω. Therefore there exists a tent $Q \subset K$ with associated mapping ψ such that $\xi_0 \in \operatorname{rint} Q$, $\psi(\xi) = \xi + o(\xi)$ and, for some $\varepsilon > 0$, $\psi(Q \cap \varepsilon B) \subset \Omega$. On the other hand, there exists an $\varepsilon' > 0$ such that, for any $\xi \in \varepsilon' B$, $|o(\xi)| \leqslant \frac{1}{2}|\xi|$ and so $|\psi(\xi)| \geqslant |\xi| - |o(\xi)| \geqslant |\xi| - \frac{1}{2}|\xi| = \frac{1}{2}|\xi|$. As we may assume $\varepsilon' \leqslant \varepsilon$, so $\psi(\xi) \in \Omega \setminus \{0\}$ for any $\xi \in (Q \cap \varepsilon' B) \setminus \{0\}$. But $\lambda\xi_0 \in (Q \cap \varepsilon' B) \setminus \{0\}$ for some $\lambda > 0$. Define $x_1 := \psi(\lambda\xi_0)$. Then $x_1 \in \Omega$ and $x_1 \neq 0$.

In the conclusion of Theorem B, we cannot replace "local tent" by "continuous local tent." For $s = 2$, a counter-example was given by V. G. Boltjanskiĭ([2]). A proof of Theorem B was proposed by B. N. Pschenichniy ([4]). However, his proof was found to be valid only for $s = 2$.

We prove Theorem B using the following results :

(i) (V. G. Boltjanskiĭ [1]) Let $K_1, \ldots, K_s$ be convex cones with common vertex at x_0 in X. Then the system $K_1, \ldots, K_s$ is inseparable if and only if

1) $\operatorname{rint} K_1 \cap \cdots \cap \operatorname{rint} K_s \neq \phi$,

2) there exists a system of vector subspaces $X_1, \ldots, X_s$ of X such that $X = \oplus_{j=1}^{s} X_j$ and $X^{(j)} := \oplus_{i \neq j} X_i \subset \operatorname{lin} K_j (j = 1, \ldots, s)$.

(ii) (B. N. Pshenichniy [4]) Let $F : \boldsymbol{R}^m \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be continuous in a neighborhood of $(0, 0)$ and $F(0, 0) = 0$. Assume that F is differentiable at $(0, 0)$ with the partial derivatives $\frac{\partial F}{\partial x}(0, 0) = 0$ and $\frac{\partial F}{\partial y}(0, 0)$ being an invertible matrix. Then there exists a mapping $y = f(x)$ defined in a neighborhood of $x = 0$ such that $F(x, f(x)) = 0$ and $\lim_{x \to 0} \frac{f(x)}{|x|} = 0$ with $f(0) = 0$.

Proof of Theorem B. We assume $x_0 = 0$ for simplicity.

For any given $\xi_0 \in \operatorname{rint} K = \cap_{j=1}^{s} \operatorname{rint} K_j$, there exist continuous tents $Q_j \subset K_j$ at 0 of Ω_j with associated mappings ψ_j. In particular, $\operatorname{lin} Q_j = \operatorname{lin} K_j$ and there exists an $\varepsilon > 0$ such that $\psi_j(Q_j \cap \varepsilon B) \subset \Omega_j (j = 1, \ldots, s)$. Let $X_1, \ldots, X_s$ be a system of vector subspaces of X associated with the system $Q_1, \ldots, Q_s$ as in (i). We denote by $P^{(j)}$ the projection mapping of X onto $X^{(j)}$, parallel to $X_j (j = 1, \ldots, s)$.

Now consider the following system of equations in $\xi, \eta_1, \ldots, \eta_s, \zeta \in X$:

$$\begin{cases} \psi_j(\xi + P^{(j)}\eta_j) - (\xi + \zeta) = 0 \ (j = 1, \ldots, s), \\ \qquad \sum_{j=1}^{s} \eta_j = 0. \end{cases}$$

This is of the form $F(x, y) = 0$ with $x := \xi$ and $y := (\eta_1, \ldots, \eta_s, \zeta)$. The mapping F is continuous in a neighborhood of $(0, 0)$ and $F(0, 0) = 0$. It is shown easily that F is differentiable at $(0, 0)$ with the partial derivatives $\frac{\partial F}{\partial x}(0, 0) = 0$ and

$$\frac{\partial F}{\partial y}(0,0)=\begin{bmatrix} P^{(1)} & & 0 & -I \\ & \ddots & & \vdots \\ 0 & & P^{(s)} & -I \\ I & \cdots & I & 0 \end{bmatrix},$$

where I is the identity mapping of X.

The linear mapping $\frac{\partial F}{\partial y}(0, 0)$ is bijective. In fact,

$$\begin{bmatrix} P^{(1)} & & 0 & -I \\ & \ddots & & \vdots \\ 0 & & P^{(s)} & -I \\ I & \cdots & I & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_s \\ w \end{bmatrix}=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

with $v_1, \dots, v_s$, $w \in X$ implies $P^{(1)}v_1 = w, \dots, P^{(s)}v_s = w$ and $v_1 + \cdots + v_s = 0$. Then $w \in \cap_{j=1}^{s} X^{(j)} = \{0\}$ and so $w = 0$. This implies $P^{(j)}v_j = 0$, that is, $v_j \in X_j (j = 1, \dots, s)$. Since $X = \oplus_{j=1}^{s} X_j$, it follows $v_j = 0 (j = 1, \dots, s)$. Thus the linear mapping $\frac{\partial F}{\partial y}(0, 0)$ is injective and so bijective.

By virtue of (ii), there exist solutions $\eta_j = \theta_j(\xi)$ $(j = 1, \dots, s)$ and $\zeta = o(\xi)$ with the stated properties. Define $\psi(\xi) := \xi + o(\xi)$.

In case $\xi_0 \neq 0$: For any $r > 0$, consider the convex cone $K(r) := \left\{\xi \in X ; (\xi \mid \xi_0) > |\xi||\xi_0| \left(1 - \frac{r^2}{2|\xi_0|^2}\right)\right\}$. Then $\left| |\xi_0| \frac{\xi}{|\xi|} - \xi_0 \right| < r$ for any $\xi \in K(r)$. Now there exists a $\delta > 0$ such that $(\xi_0 + \delta B) \cap \operatorname{lin} Q_j \subset Q_j (j = 1, \dots, s)$. Therefore, for any $\xi \in K(\delta) \cap \operatorname{lin} Q_j$, $|\xi_0| \frac{\xi}{|\xi|} \in (\xi_0 + \delta B) \cap \operatorname{lin} Q_j \subset Q_j$ and so $\xi \in Q_j$. Consider the convex cone $Q := \cap_{j=1}^{s} Q_j \cap K\left(\frac{\delta}{2}\right)$ and the mappings $\varphi_j(\xi) := \xi + P^{(j)}\theta_j(\xi) (j = 1, \dots, s)$. By the property of θ_j, there exists an $\varepsilon' > 0$ such that $\varphi_j(Q \cap \varepsilon' B) \subset \varepsilon B \cap K(\delta)$ $(j = 1, \dots, s)$. On the other hand, $\varphi_j(Q \cap \varepsilon' B) \subset \operatorname{lin} Q + X^{(j)} \subset \operatorname{lin} Q_j$. Therefore $\varphi_j(Q \cap \varepsilon' B) \subset \varepsilon B \cap \operatorname{lin} Q_j \cap K(\delta) \subset Q_j \cap \varepsilon B$ and so $\psi_j(\varphi_j(Q \cap \varepsilon' B)) \subset \Omega$. As $\psi(\xi) = \psi_j(\varphi_j(\xi))$ $(j = 1, \dots, s)$, so we have $\psi(Q \cap \varepsilon' B) \subset \Omega$. Thus K is a local tent at 0 of Ω.

In case $\xi_0 = 0$: There exists a $\delta > 0$ such that $\delta B \cap \operatorname{lin} Q_j \subset Q_j (j = 1, \dots, s)$. Therefore $Q_j = \operatorname{lin} Q_j$. Let $Q := \cap_{j=1}^{s} Q_j$. Then, for some $\varepsilon' > 0$, $\varphi_j(\varepsilon' B) \subset \varepsilon B$ and so $\varphi_j(Q \cap \varepsilon' B) \subset Q_j \cap \varepsilon B$, where $\varphi_j(\xi) := \xi + P^{(j)}\theta_j(\xi)$ as before. Therefore we have $\psi(Q \cap \varepsilon' B) \subset \Omega$ as in case $\xi_0 \neq 0$. Thus K is a local tent at 0 of Ω. Q.E.D.

References

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[3] ——: The method of tents in the theory of extremal problems. Uspehi Mat. Nauk, **30**, 3–35 (1975); Russian Math. Surveys, **30**: 3, 1–54 (1975).

[4] Pschenichniy, B. N.: Convex Analysis and Extremal Problems. Nauka (1980).