70. *Explicit Formulas and Asymptotic Expansions for Certain Mean Square of Hurwitz Zeta-functions*

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Let $\zeta(s, \alpha)$ be the Hurwitz zeta-function with a positive parameter α, and $\zeta_1(s, \alpha) = \zeta(s, \alpha) - \alpha^{-s}$. Recently, three proofs of the conjecture

$$\int_0^1 \left|\zeta_1\left(\frac{1}{2}+it, \alpha\right)\right|^2 d\alpha = \log(t/2\pi) + \gamma + O(t^{-\frac{1}{4}}), \tag{1}$$

where $t \geq 1$ and γ is Euler's constant, have appeared. Zhang's proof [10] is based on the functional equation of $\zeta(s, \alpha)$, and actually, he proved the following stronger result:

$$\int_0^1 \left|\zeta_1\left(\frac{1}{2}+it, \alpha\right)\right|^2 d\alpha = \log(t/2\pi) + \gamma - 2\mathrm{Re}\frac{\zeta\left(\frac{1}{2}+it\right)}{\frac{1}{2}+it} + O(t^{-1}), \tag{2}$$

where $\zeta(s)$ is the Riemann zeta-function. Another proof of (2) is given in Andersson [1], who obtained certain explicit formulas (Corollaries 1 and 2 below) which implies (2). His proof is based on Mikolás' idea [7] of using Parseval's identity. The third proof, sketched in the authors' article [6], is a variant of Atkinson's method, and the key lemma is the explicit formula [6, (3.1)]. The main idea of this proof is based on the works of Atkinson [2], Motohashi [8] and the authors [4].

By refining the argument of the third proof, we can prove several explicit formulas and asymptotic expansions, which we announce in this note. The proofs will appear elsewhere.

The first result is a further refinement of Andersson-Zhang's formula (2). Let $\Gamma(s)$ be the gamma-function, and $\psi(s) = (\Gamma'/\Gamma)(s)$. Then,

Theorem 1. *For any integer $K \geq 0$, we have the asymptotic expansion*

$$\int_0^1 \left|\zeta_1\left(\frac{1}{2}+it, \alpha\right)\right|^2 d\alpha$$
$$= \gamma - \log 2\pi + \mathrm{Re}\,\psi\left(\frac{1}{2}+it\right) - 2\,\mathrm{Re}\frac{\zeta\left(\frac{1}{2}+it\right)-1}{\frac{1}{2}+it}$$
$$- 2\,\mathrm{Re}\sum_{k=1}^{K}\frac{(-1)^{k-1}(k-1)!}{\left(\frac{3}{2}-k+it\right)\left(\frac{5}{2}-k+it\right)\cdots\left(\frac{1}{2}+it\right)}\sum_{l=1}^{\infty} l^{-k}(l+1)^{-\frac{3}{2}+k-it}$$
$$+ O(t^{-K-1}).$$

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Remark 1. The empty sum is to be considered as zero.

Remark 2. Since $\mathrm{Re}\,\psi\left(\frac{1}{2}+it\right)=\log t+O(t^{-2})$, Theorem 1 implies (2).

Theorem 1 can be obtained, by taking the limit $\sigma\to\frac{1}{2}$, from the following

Theorem 2. *If* $0<\sigma<2$, $\sigma\notin\left\{\frac{1}{2},1\right\}$ *and* $t\geq 1$, *then*

$$\int_0^1 |\zeta_1(\sigma+it,\alpha)|^2 d\alpha$$

$$=\frac{1}{2\sigma-1}+2\Gamma(2\sigma-1)\zeta(2\sigma-1)\,\mathrm{Re}\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}-2\,\mathrm{Re}\frac{\zeta(\sigma+it)-1}{1-\sigma+it}$$

$$-2\,\mathrm{Re}\sum_{k=1}^{K}(-1)^{k-1}\frac{(2-2\sigma)_{k-1}(\sigma+it)_{1-k}}{1-\sigma+it}\sum_{l=1}^{\infty}l^{-k}(l+1)^{-\sigma-1+k-it}$$

$$+O(t^{-K-1})$$

for any integer $K\geq 0$, *where* $(s)_n=\Gamma(s+n)/\Gamma(s)$ *for any integer* n.

The asymptotic formula on the line $\sigma=1$ can also be deduced from Theorem 2, as the limit case $\sigma\to 1$.

More generally, starting from [6, (3.1)], we can prove the following explicit formula. Let u, v be complex variables, and E the set of (u,v) at which some factor in (3) below has a singularity.

Theorem 3. *Let* $N\geq 1$ *be an integer,* $-N+1<\mathrm{Re}\,u<N+1$, $-N+1<\mathrm{Re}\,v<N+1$, *and* $(u,v)\notin E$. *Then it holds*

$$(3)\quad \int_0^1 \zeta_1(u,\alpha)\zeta_1(v,\alpha)d\alpha$$

$$=\frac{1}{u+v-1}+\Gamma(u+v-1)\zeta(u+v-1)\left(\frac{\Gamma(1-v)}{\Gamma(u)}+\frac{\Gamma(1-u)}{\Gamma(v)}\right)$$

$$-S_N(u,v)-S_N(v,u)-T_N(u,v)-T_N(v,u),$$

where

$$S_N(u,v)=\sum_{n=0}^{N-1}\frac{(u)_n}{(1-v)_{n+1}}(\zeta(u+n)-1),$$

$$T_N(u,v)=\frac{(u)_N}{(1-v)_N}\sum_{l=1}^{\infty}l^{1-u-v}\int_l^{\infty}\beta^{u+v-2}(1+\beta)^{-u-N}d\beta.$$

Moreover, $T_N(u,v)$ *has the expression*

$$T_N(u,v)=\sum_{k=1}^{K}(-1)^{k-1}\frac{(2-u-v)_{k-1}(u)_{N-k}}{(1-v)_N}\sum_{l=1}^{\infty}l^{-k}(l+1)^{-u-N+k}$$

$$+(-1)^K\frac{(2-u-v)_K(u)_{N-K}}{(1-v)_N}\sum_{l=1}^{\infty}l^{1-u-v}\int_l^{\infty}\beta^{u+v-K-2}(1+\beta)^{-u-N+K}d\beta$$

for any integer $K\geq 0$.

Taking the limit $N\to\infty$ in Theorem 3, we have the following explicit result, because $T_N(u,v)\to 0$ as $N\to\infty$.

Corollary 1. *Let u, v be as in Theorem 3. Then*

$$\int_0^1 \zeta_1(u, \alpha)\zeta_1(v, \alpha)d\alpha$$
$$= \frac{1}{u+v-1} + \Gamma(u+v-1)\zeta(u+v-1)\left(\frac{\Gamma(1-v)}{\Gamma(u)} + \frac{\Gamma(1-u)}{\Gamma(v)}\right)$$
$$- \sum_{n=0}^{\infty} \frac{(u)_n}{(1-v)_{n+1}}(\zeta(u+n)-1) - \sum_{n=0}^{\infty} \frac{(v)_n}{(1-u)_{n+1}}(\zeta(v+n)-1).$$

Taking the limit $u \to \frac{1}{2} + it$ and $v \to \frac{1}{2} - it$ in Corollary 1, we have another refinement of (2):

Corollary 2.

$$\int_0^1 \left|\zeta_1\left(\frac{1}{2}+it, \alpha\right)\right|^2 d\alpha = \gamma - \log 2\pi + \operatorname{Re}\psi\left(\frac{1}{2}+it\right)$$
$$- 2\operatorname{Re}\sum_{n=0}^{\infty} \frac{\zeta\left(\frac{1}{2}+n+it\right)-1}{\frac{1}{2}+n+it}.$$

We note that Andersson [1] has shown Corollaries 1 and 2 by a different method. The special case $t=0$ in Corollary 2 is also given in Zhang [10, Theorem 3].

Next, taking $u = \sigma + it$ and $v = \sigma - it$ in Theorem 3, we obtain the following generalization of Theorem 2.

Corollary 3. *Let N, K be integers with $N \geq 1$ and $K \geq 0$. Then, for any σ satisfying $-N+1 < \sigma < N+1$, $2\sigma - 1 \notin \{1, 0, -1, -2, \ldots\}$ and any $t \geq 1$, we have*

$$\int_0^1 |\zeta_1(\sigma+it, \alpha)|^2 d\alpha$$
$$= \frac{1}{2\sigma-1} + 2\Gamma(2\sigma-1)\zeta(2\sigma-1)\operatorname{Re}\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}$$
$$- 2\operatorname{Re}\sum_{n=0}^{N-1} \frac{(\sigma+it)_n}{(1-\sigma+it)_{n+1}}(\zeta(\sigma+it+n)-1)$$
$$- 2\operatorname{Re}\sum_{k=1}^{K} (-1)^{k-1}\frac{(2-2\sigma)_{k-1}(\sigma+it)_{N-k}}{(1-\sigma+it)_N}\sum_{l=1}^{\infty} l^{-k}(l+1)^{-\sigma-N+k-it}$$
$$+ O(t^{-K-1}).$$

The exceptional cases $(u, v) \in E$ in Theorem 3 can be treated as the limit cases. Theorem 1 is such an example. Another example is

Corollary 4. *For any integer $m \geq 0$, we have*

$$\int_0^1 \zeta_1(-m, \alpha)^2 d\alpha$$
$$= -\frac{1}{2m+1} + (-1)^{m+1}\frac{(m!)^2}{(2m+1)!}\zeta(-2m-1)$$
$$- 2(-1)^m \frac{(m!)^2}{(2m+2)!} - 2\sum_{n=0}^{m} \frac{(-1)^n(m!)^2}{(m+n+1)!(m-n)!}(\zeta(n-m)-1).$$

It is also possible to deduce the asymptotic formula at positive integers.

The closed form of Corollary 4 can also be proved from the well-known formula $\zeta(-m, \alpha) = -B_{m+1}(\alpha)/(m+1)$, where $B_{m+1}(\alpha)$ is the $(m+1)$-th Bernoulli polynomial. This is achieved by the method similar to that described in Section 4 of [5].

It should be mentioned that the results of Mikolás [7, Satz 3] can be treated in the frame of our method. The formula

$$\begin{aligned} (4)\qquad & \zeta(u, \alpha)\zeta(v, \alpha) \\ &= \zeta(u+v, \alpha) + \Gamma(u+v-1)\zeta(u+v-1)\left(\frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)}\right) \\ &\quad + g(u, v; \alpha) + g(v, u; \alpha) \end{aligned}$$

holds for $\mathrm{Re}\, u < 1$, $\mathrm{Re}\, v < 1$, which is equivalent to [6, (3.1)]. Taking $u = \sigma + it$, $v = \sigma - it$, with $\sigma < \frac{1}{2}$ in (4), and integrating both sides, we obtain Mikolás' result [7, (5.4)]. Moreover, it can also be proved as the limit case $q \to \infty$ of our discrete mean value result [6, Theorem 2].

Finally we consider the mean square of the derivative $\zeta_1'(s, \alpha) = \frac{\partial}{\partial s}\zeta_1(s, \alpha)$. Zhang [9] proved, among other things, that there exist constants A and B, for which

$$\begin{aligned} (5)\qquad \int_0^1 \left| \zeta_1'\left(\frac{1}{2} + it, \alpha\right)\right|^2 d\alpha &= \frac{1}{3}\log^3(t/2\pi) \\ &+ \gamma \log^2(t/2\pi) - 2B\log(t/2\pi) + A + \rho(t) \end{aligned}$$

holds, where $\rho(t) = O(t^{-1/6}(\log t)^{10/3})$. Zhang defined A and B as certain integrals, but it can be seen that $A = 2\gamma_2$ and $B = -\gamma_1$, where γ_1 and γ_2 are generalized Euler's constants defined by

$$\zeta(1+s) = s^{-1} + \gamma + \gamma_1 s + \gamma_2 s^2 + \cdots.$$

On the other hand, as was first noticed in Katsurada [3], our method can be applied to the mean square of derivatives (of any order) of zeta and L-functions. Here we do not state the rather complicated form of the general result, but as a special case, we write down the following expression of $\rho(t)$ in (5).

Theorem 4. *We have*

$$\rho(t) = -2\,\mathrm{Re}\frac{\zeta'\left(\frac{1}{2}+it\right)}{\left(\frac{1}{2}+it\right)^2} - 2\,\mathrm{Re}\frac{\partial^2}{\partial u \partial v}T_1(u, v)\Big|_{u=\frac{1}{2}+it, v=\frac{1}{2}-it} + O(t^{-2}\log^2 2t).$$

This in particular implies $\rho(t) = O(t^{-1})$, which improves Zhang's estimate of $\rho(t)$ mentioned above.

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