# 69. *On Certain Infinite Series of Dirichlet Type*

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**1. Introduction.** Let $\alpha$ denote a fixed real number and put $M=\left[\alpha+\frac{1}{2}\right]+1$, where $[x]$ denotes the integral part of $x$. Let $h(z)$ be a complex valued function which is regular and non-vanishing in the half plane $\mathrm{Re}(z)>\alpha$. In this paper, we shall consider the infinite series of the form

$$\psi_h(s,a)=\sum_{n=M}^{\infty}\frac{e(na)}{h(n)^s},$$

where $e(w)$ denotes an abbreviation of $\exp(2\pi iw)$ and $a$ is a real number with $0<a<1$. Here and in what follows, $h(z)^s=\exp(s\log h(z))$ with a fixed branch of $\log h(z)$. Moreover, we impose the following conditions on $h(z)$:

(A.1) $\psi_h(s,a)$ converges for all sufficiently large real values of $s$.

(A.2) $\log|h(z)|\ll\log|z|$ and $\arg h(z)\ll\log|z|$ for $|z|\gg 0$, where $\arg h(z)$ denotes the argument of $h(z)$.

(A.3) $|h(z)|\to\infty$ as $|z|\to\infty$.

Then we obtain

**Theorem 1.** *Under the above assumptions, $\psi_h(s,a)$ is extended to an integral function of $s$ in the whole complex $s$-plane.*

**Example 1.** Let $h(z)$ be a non-constant polynomial of $z$ with complex coefficients. Take an integer $M$ such that $h(z)$ has no zeros in $\mathrm{Re}(z)>M-1$ and $\alpha=M-1$. Then $\psi_h(s,a)$ is absolutely convergent for $s>1$ and for any fixed branch of $\log h(z)$. Hence, by Theorem 1, $\psi_h(s,a)$ can be continued analytically to an integral function of $s$.

**Example 2.** Let $g(x,y)$ be a polynomial in $x$ and $y$ with complex coefficients. Suppose that the degree of $g(x,y)$ in $x$ is at least 1. Take a positive integer $M$ such that $g(z,\log z)$ has no zeros in $\mathrm{Re}(z)>M-1$, where any fixed branch is taken for the logarithm. If we take $h(z)=g(z,\log z)$ and $\alpha=M-1$, then $\psi_h(s,a)$ is absolutely convergent for $s>1$ and for any fixed branch of $\log h(z)$. Therefore, by Theorem 1, $\psi_h(s,a)$ is extended to an integral function of $s$.

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**2. Proof of Theorem 1.** The method of the proof is similar to that of Theorem 1 in [2].

Let $C$ be the rectangle in the $z$-plane consisting of the line segments $C_1$, $C_2$, $C_3$ and $C_4$, joining $\xi-Ni$, $\left(N+\frac{1}{2}\right)-Ni$, $\left(N+\frac{1}{2}\right)+Ni$, $\xi+Ni$ and $\xi-Ni$, where $\xi=M-\frac{1}{2}$ and $N$ is a sufficiently large integer. Consider the integral

$$I(s)=\int_C f(s,z)\,dz,$$

where

$$f(s,z)=\frac{e(az)}{e(z)-1}h(z)^{-s}$$

and $s$ is a sufficiently large real number. By the residue theorem, we have

(1) $$I(s)=\sum_{n=M}^{N}\frac{e(na)}{h(n)^s}.$$

On the other hand, we see that

(2) $$I(s)=\Big(\int_{C_1}+\int_{C_2}+\int_{C_3}+\int_{C_4}\Big)f(s,z)\,dz$$
$$=I_1+I_2+I_3+I_4,\ \text{say.}$$

Since $s>1$ and $|h(z)^{-s}|=|h(z)|^{-s}$, we can find a number $N_0$ depending on $\varepsilon$ such that

$$|h(x-Ni)^{-s}|<\varepsilon\ (x\geqq\xi,\ N>N_0)$$

for any given $\varepsilon>0$ in view of (A.3). So we have

$$|I_1|<\frac{\varepsilon e^{2\pi aN}}{e^{2\pi N}-1}\int_{\xi}^{N+\frac{1}{2}}dx<\varepsilon,$$

because $0<a<1$. This implies that $I_1\to 0$ as $N\to\infty$. Similarly, $I_2$, $I_3\to 0$ as $N\to\infty$. By letting $N\to\infty$, we infer from (1), (2) and (A.1) that

(3) $$\phi_h(s,a)=ie(a\xi)\Big[\int_0^\infty\frac{e^{2\pi(1-a)x}}{e^{2\pi x}+1}h(\xi+ix)^{-s}\,dx$$
$$+\int_0^\infty\frac{e^{2\pi ax}}{e^{2\pi x}+1}h(\xi-ix)^{-s}\,dx\Big].$$

This formula holds for all sufficiently large real values of $s$. Studying the behavior of the above integrals in the whole plane of the complex variable $s$, we see from (A.2) that both integrals of (3) converge uniformly in any finite region of the complex $s$-plane and so define integral functions of $s$. This completes the proof of Theorem 1.

**3. On the values of $\phi_h(s,a)$ at non-positive integers.** We start with introducing the $\beta$-function defined by

$$\beta(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+x}.$$

It is known (cf. [1], p. 523) that

(4) $$\beta(x)+\beta(1-x)=\frac{\pi}{\sin\pi x}.$$

**Theorem 2.** $\phi_h(0,a)=\frac{ie(a\xi)}{2\sin(a\pi)}$, *where* $\xi=M-\frac{1}{2}$.

*Proof.* By (3), we have

(5) $$\phi_h(0,a)=ie(a\xi)\Big[\int_0^\infty\frac{e^{2\pi(1-a)x}}{e^{2\pi x}+1}dx+\int_0^\infty\frac{e^{2\pi ax}}{e^{2\pi x}+1}dx\Big]$$
$$=ie(a\xi)[J_1+J_2],\ \text{say.}$$

As is easily verified,

$$\begin{aligned} J_1 &= \int_0^\infty e^{-2\pi ax} \sum_{n=0}^\infty (-1)^n e^{-2\pi nx} dx \\ &= \sum_{n=0}^\infty (-1)^n \int_0^\infty e^{-2\pi(n+a)x} dx \\ &= \frac{1}{2\pi} \sum_{n=0}^\infty \frac{(-1)^n}{n+a} \\ &= \frac{\beta(a)}{2\pi}. \end{aligned}$$

Similarly, we get

$$J_2 = \frac{\beta(1-a)}{2\pi}.$$

Then our assertion follows immediately from (4) and (5).

Put

$$E(x) = \frac{\pi}{\sin \pi x}.$$

Let $m$ be a non-negative integer and $E^{(m)}(x)$ the $m$-th derivative of $E(x)$. Then we have

**Lemma.**

$$E^{(m)}(x) = \pi^{m+1} \frac{g_m(\cos \pi x)}{(\sin \pi x)^{m+1}},$$

*where* $g_m(\cos \pi x)$ *is a linear combination of* $(\cos \pi x)^{2j}\left(0 \leqq j \leq \frac{m}{2}\right)$ *or* $(\cos \pi x)^{2j+1}\left(0 \leqq j \leq \left[\frac{m}{2}\right]\right)$ *with rational integer coefficients according as* $m$ *is even or odd.*

*Proof.* The lemma is easily shown by induction on $m$. So we omit the proof of it.

In the following, let $h(z)$, $M$ and $\psi_h(s, a)$ be as in Example 1. Let $\boldsymbol{F}$ be a subfield of the complex number field. Suppose that all coefficients of $h(z)$ are contained in $\boldsymbol{F}$. Then we obtain

**Theorem 3.** *The value* $\psi_h(-m, a)$ *belongs to the field* $\boldsymbol{F}(i\cot a\pi)$ *for any non-negative integer* $m$.

*Proof.* By (3), $\psi_h(-m, a)$ is a linear combination of $J_k(a)$ $(k = 0, 1, \dots, md)$ with coefficients in $\boldsymbol{F}$, where $d$ is the degree of $h(z)$ and

$$J_k(a) = ie(a\xi) \left[\int_0^\infty \frac{e^{2\pi(1-a)x}}{e^{2\pi x}+1} (ix)^k dx + \int_0^\infty \frac{e^{2\pi ax}}{e^{2\pi x}+1} (-ix)^k dx\right].$$

By noting that

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx \ (s > 0),$$

it is not difficult to see that

$$\begin{aligned} J_k(a) = i^{k+1} e(a\xi) \Bigg[&\sum_{n=0}^\infty (-1)^n \int_0^\infty e^{-2\pi(n+a)x} x^k dx \\ &+ (-1)^k \sum_{n=0}^\infty (-1)^n \int_0^\infty e^{-2\pi(n+1-a)x} x^k dx\Bigg] \end{aligned}$$

$$= \frac{i^{k+1}e(a\xi)\Gamma(k+1)}{(2\pi)^{k+1}}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+a)^{k+1}} + (-1)^k\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1-a)^{k+1}}\right]$$

$$= \frac{(-1)^k i^{k+1}e(a\xi)}{(2\pi)^{k+1}}(\beta^{(k)}(a) + (-1)^k\beta^{(k)}(1-a)),$$

so that from (4) we get

$$J_k(a) = \frac{(-1)^k i^{k+1}e(a\xi)}{(2\pi)^{k+1}}E^{(k)}(a).$$

We remark that $e(a\xi) = e^{2\pi iaM}e^{-\pi ia}$,

$$e^{2\pi iaM} = (-1)^M\left[\frac{(1-i\cot a\pi)^2}{1-(i\cot a\pi)^2}\right]^M$$

and $$(\sin a\pi)^2 = \frac{1}{1-(i\cot a\pi)^2}.$$

Thus if $k$ is even, then, by the lemma, $J_k(a)$ is an element of $\boldsymbol{F}(i\cot a\pi)$, because $\frac{ie^{-\pi ia}}{\sin a\pi} = 1 + i\cot a\pi$. Similarly, if $k$ is odd, then $J_k(a)$ is an element of $\boldsymbol{F}(i\cot a\pi)$, because $e^{-\pi ia}\cos a\pi = \frac{i\cot a\pi}{i\cot a\pi - 1}$. This completes the proof.

## References

[1] T. J. I'A. Bromwich: An Introduction to the Theory of Infinite Series. Macmillan and Co., Ltd., London (1926).

[2] M. Toyoizumi: On certain infinite series (to appear in Tokyo J. Math.).