# 68. *On the Existence of Characters of the Schur Index 2 of the Simple Finite Steinberg Groups of Type* $(^2E_6)$*)

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Let $\chi$ be a complex irreducible character of a finite group and $k$ be a field of characteristic 0. Then we denote by $m_k(\chi)$ the Schur index of $\chi$ with respect to $k$.

It has been known that the simple group $\mathrm{PSU}(3, q^2)$ has an irreducible character $\chi$ with $m_{\boldsymbol{Q}}(\chi) = 2$ (R. Gow [4]). In [5], (7.6), G. Lusztig found that $\mathrm{PSU}(3, q^2)$ or $\mathrm{PSU}(6, q^2)$ has a rational-valued irreducible character $\chi$ such that $m_{\boldsymbol{Q}}(\chi) = m_{\boldsymbol{R}}(\chi) = m_{\boldsymbol{Q}_p}(\chi) = 2$ ($q$ is a power of $p$) and $m_{\boldsymbol{Q}_l}(\chi) = 1$ for any prime number $l \neq p$. For $\mathrm{PSU}(3, q^2)$, this $\chi$ coincides with the one described above. In this note we shall show that the simple finite Steinberg group ${}^2E_6(q^2)$ has (at least) two rational-valued irreducible characters $\chi$ such that $m_{\boldsymbol{Q}}(\chi) = m_{\boldsymbol{R}}(\chi) = m_{\boldsymbol{Q}_p}(\chi) = 2$ and $m_{\boldsymbol{Q}_l}(\chi) = 1$ for any prime number $l \neq p$. This will follow from Lusztig's classification theory of the unipotent representations of finite groups of Lie type (see [2], pp. 480–481).

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Let $\boldsymbol{F}_q$ be a finite field with $q$ elements, of characteristic $p$. If $X$ is an algebraic group defined over $\boldsymbol{F}_q$, then $X(q)$ denotes the group of $\boldsymbol{F}_q$-rational points of $X$. Then we have

**Lemma.** *Let $M$ be a connected, reductive algebraic group, defined over $\boldsymbol{F}_q$, whose Coxeter graph is of type $(^2A_2)$ or $(^2A_5)$. Let $R$ be a (unique) cuspidal unipotent representation of $M(q)$, with the character $\chi$. Then $\chi$ is rational-valued and we have $m_{\boldsymbol{R}}(\chi) = m_{\boldsymbol{Q}_p}(\chi) = 2$ and $m_{\boldsymbol{Q}_l}(\chi) = 1$ for any prime number $l \neq p$.*

This is stated in [5] as (7.6) without detailed proof. We shall now sketch the proof. Let $X_f$ be as in [5], (1.7). Let $l$ be any prime number $\neq p$. For $i \geq 0$, put $H_c^i(X_f) = H_c^i(X_f, \bar{\boldsymbol{Q}}_l) = H_c^i(X_f, \boldsymbol{Q}_l) \otimes \bar{\boldsymbol{Q}}_l$, where $\bar{\boldsymbol{Q}}_l$ is an algebraic closure of $\boldsymbol{Q}_l$. Then $H_c^i(X_f)$ is a $\bar{\boldsymbol{Q}}_l[M(q)]$-module defined over $\boldsymbol{Q}_l$. Let $F: M \to M$ be the Frobenius map. Then $F^2$ acts on $H_c^i(X_f)$. Let $r$ be the semisimple rank of $M$. Let $V$ be the $F^2$-eigensubspace of $H_c^r(X_f)$ corresponding to the eigenvalue $-q$ (resp. $-q^3$) if $r = 2$ (resp. if $r = 5$). Then $V$ is an irreducible $M(q)$-module and is isomorphic to $R$. As $H_c^r(X_f)$ is defined over $\boldsymbol{Q}_l$ and $\langle R, H_c^r(X_f)\rangle_{M(q)} = 1$, we have $m_{\boldsymbol{Q}_l}(\chi) = 1$. Since $\langle H_c^i(X_f), H_c^j(X_f)\rangle_{M(q)} = 0$ if $i \neq j$, the character of the virtual module $W = \sum(-1)^i H_c^i(X_f)$ is rational-valued and each irreducible component of $W$ has a different degree, $\chi$ is rational-valued (see below). By [5], (4.4), there is a $M(q)$-equivariant antisymmetric bilinear form on $V$. As $\boldsymbol{Q}_l \simeq \boldsymbol{C}$, $V$ may be

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regarded as a $\boldsymbol{C}[M(q)]$-module. Hence, by a theorem of Frobenius-Schur, we have $m_{\boldsymbol{R}}(\chi)=2$. And, by Hasse's sum formula, we have $m_{\boldsymbol{Q}_p}(\chi)=2$.

Now let $G$ be a connected, reductive algbraic group, defined over $\boldsymbol{F}_q$, whose Coxeter graph is of type $({}^2E_6)$. Let $P$ be a parabolic subgroup of $G$, defined over $\boldsymbol{F}_q$, which has a Levi part $L$ (over $\boldsymbol{F}_q$) of type $({}^2A_5)$. Let $\rho$ be the unique cuspidal unipotent character of $L(q)$ (see [2], 13.7, p. 457). Then, by the lemma above, $\rho$ is rational-valued and $m_{\boldsymbol{Q}}(\rho)=m_{\boldsymbol{R}}(\rho)=m_{\boldsymbol{Q}_p}(\rho)=2$ and $m_{\boldsymbol{Q}_l}(\rho)=1$ for any prime number $l\neq p$. Let $P\rightarrow L$ be the natural map and put $\tilde{\rho}=\rho\, o(P(q)\rightarrow L(q))$. Then the character $\tilde{\rho}$ of $P(q)$ has the rationality similar to that of $\rho$. Let $R$ be a representation of $P(q)$ which affords $\tilde{\rho}$. Then we find that $\mathrm{End}_{G(q)}(\mathrm{Ind}_{P(q)}^{G(q)}(R))$ is isomorphic to the group algebra of the Weyl group $W$ of type $(A_1)$ (cf. [6], Table II, p. 35). Thus, 1 and $\varepsilon$ being the irreducible characters of $W$, we have

$$\tilde{\rho}^{G(q)}=\mathrm{Ind}_{P(q)}^{G(q)}(\tilde{\rho})=\rho_1+\rho_2,$$

where $\rho_1$ (resp. $\rho_2$) is an irreducible character of $G(q)$ corresponding, for instance, to 1 (resp. to $\varepsilon$).

We first show that $\rho_1$ and $\rho_2$ are rational-valued. In fact, let $\bar{\boldsymbol{Q}}$ be an algebraic closure of $\boldsymbol{Q}$ and let $\sigma$ be any element of $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$. As $\rho$ is rational-valued, $\tilde{\rho}^{G(q)}$ is also rational-valued, so that we have:

$$\rho_1+\rho_2=\tilde{\rho}^{G(q)}=(\tilde{\rho}^{G(q)})^{\sigma}=\rho_1^{\sigma}+\rho_2^{\sigma}.$$

But as $\rho_1(1)\neq\rho_2(1)$ (see [2], p. 481), we must have $\rho_i^{\sigma}=\rho_i$, $i=1,2$. Thus $\rho_1$ and $\rho_2$ are rational-valued.

We next show that, for $i=1,2$, we have $m_{\boldsymbol{Q}}(\rho_i)=m_{\boldsymbol{R}}(\rho_i)=m_{\boldsymbol{Q}_p}(\rho_i)=2$ and $m_{\boldsymbol{Q}_l}(\rho_i)=1$ for any prime number $l\neq p$. If $l$ is any prime number $\neq p$, then $\tilde{\rho}^{G(q)}$ is realizable in $\boldsymbol{Q}_l$, so that, by a theorem of Schur, we have $m_{\boldsymbol{Q}_l}(\rho_i)=1$ for $i=1,2$. Suppose that $m_{\boldsymbol{Q}_p}(\rho_i)=1$ ($i=1$ or 2). Then $\rho_i$ is realizable in $\boldsymbol{Q}_p$, so that, by Schur's theorem, $2=m_{\boldsymbol{Q}_p}(\tilde{\rho})$ must divide $\langle\tilde{\rho},\rho_i\,|\,P(q)\rangle_{P(q)}=\langle\tilde{\rho}^{G(q)},\rho_i\rangle_{G(q)}=1$, a contradiction. Therefore we must have $m_{\boldsymbol{Q}_p}(\rho_i)=2\,(i=1,2)$. [We note that, by the Brauer-Speiser theorem, if $\chi$ is a rational-valued irreducible character of a finite group, $m_{\boldsymbol{Q}}(\chi)$ is at most two.] Similarly we have $m_{\boldsymbol{R}}(\rho_i)=2\,(i=1,2)$.

Now let $G$ be as above and assume that $G$ is a simply-connected semisimple group. Let $Z$ be the centre of $G$. Then, in view of [3], Proposition 7.10, we see that $\rho_1$ and $\rho_2$ are trivial on $Z(q)$, so that they may be regarded as characters of $G(q)/Z(q)={}^2E_6(q^2)$. Thus we get:

**Theorem.** *The simple Steinberg group ${}^2E_6(q^2)$ has (at least) two rational-valued irreducible characters $\chi_1,\chi_2$ such that, for $i=1,2$, we have $m_{\boldsymbol{Q}}(\chi_i)=m_{\boldsymbol{R}}(\chi_i)=m_{\boldsymbol{Q}_p}(\chi_i)=2$ and $m_{\boldsymbol{Q}_l}(\chi_i)=1$ for any prime number $l\neq p$.*

As to the rationality of other unipotent characters of $G(q)$ or ${}^2E_6(q^2)$, we see, by a result of C. T. Benson and C. W. Curtis [1], that all the principal series unipotent characters are realizable in $\boldsymbol{Q}$.

The argument in this note may be applied to the groups of type $({}^2A_r)$. In fact, we see that, for each $n\neq 2,4$, $\mathrm{PSU}(n,q^2)$ has rational-valued irreducible characters $\chi$ such that $m_{\boldsymbol{Q}}(\chi)=m_{\boldsymbol{R}}(\chi)=m_{\boldsymbol{Q}_p}(\chi)=2$ and

$m_{Q_l}(\chi) = 1$ for any prime number $l \neq p$.

## References


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