64. *On the Asymptotic Formula for the Number of Representations of Numbers as the Sum of a Prime and a k-th Power*

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§1. For an integer $k \geq 2$, let $E_k(X)$ be the number of natural numbers $n \leq X$ such that n is not representable as the sum of a prime and a k-th power. In 1937, Davenport and Heilbronn [3] proved that $E_k(X) = O(X(\log X)^{-c_k})$ with a positive constant c_k depending only on k, in other words, almost all natural numbers are representable as the sum of a prime and a k-th power. After their result, some articles established sharper bounds for $E_k(X)$, and, at present, the best result is $E_k(X) = O(X^{1-\delta_k})$ with a positive constant δ_k depending only on k, which was proved by A. I. Vinogradov [9] and Brünner, Perelli, and Pintz [1] for $k = 2$, and by Plaksin [7] and Zaccagnini [10] for $k \geq 3$. On the difference of the situations between the cases $k = 2$ and $k \geq 3$, we relate in §4 briefly.

On the other hand, let $R_k(n)$ be the number of representations of n as the sum of a prime and a k-th power, $\rho_n(d) = \rho_{n,k}(d)$ be the number of solutions m of the congruence $m^k - n \equiv 0 \pmod d$ with $1 \leq m \leq d$, and let I_k be the set of all natural numbers n such that the polynomial $x^k - n$ is irreducible in $\boldsymbol{Q}[x]$, where $\boldsymbol{Q}$ is the rational number field. As for the asymptotic behavior of $R_k(n)$, it is conjectured that

$$R_k(n) \sim \mathfrak{S}_k(n) \frac{n^{1/k}}{\log n},$$

as n tends to the infinity, providing $n \in I_k$, where

$$\mathfrak{S}_k(n) = \prod_p \left(1 - \frac{\rho_n(p) - 1}{p - 1}\right),$$

and hereafter the letter p stands for prime numbers. For $k = 2$, this was conjectured by Hardy and Littlewood [4, Conjecture H], and Miech [6] proved that

$$R_2(n) = \mathfrak{S}_2(n) \frac{\sqrt{n}}{\log n} \left(1 + O\left(\frac{\log\log n}{\log n}\right)\right)$$

for all but $O(X(\log X)^{-A})$ natural numbers $n \leq X$ with any fixed $A > 0$. For each $k \geq 3$, we can also establish an asymptotic formula for $R_k(n)$ valid for almost all n:

Theorem. *For a fixed integer $k \geq 3$, and for any fixed $A > 0$, we have*

$$R_k(n) = \mathfrak{S}_k(n) \frac{n^{1/k}}{\log n} \left(1 + O\left(\frac{\log\log n}{\log n}\right)\right) \tag{1}$$

for $n \leq X$ with at most $O(X(\log X)^{-A})$ exceptions.

Because of the possible existence of the Siegel zeros, Miech's result and our result seem the best possible for the present. The proof of our Theorem is described in [5] in detail.

§2. Our proof is in the frame work of the circle method of Hardy and Littlewood, as in the articles cited in the preceding section. The most important part of our argument is the treatment of the sum $\mathfrak{S}_k(n, Q)$, introduced below, which is the singular series in our problem. In the articles [3], [7] and [10], the singular series $\mathfrak{S}_k(n, Q)$ is approximated, for almost all n, by a finite product of the form $\Pi_{p \leq P}(1 - (\rho_n(p) - 1)/(p - 1))$ with a suitable parameter P. In contrast with this, we shall approximate $\mathfrak{S}_k(n, Q)$ by an infinite product $\mathfrak{S}_k(n)$ for almost all n.

Let $A > 0$ be any fixed constant, B be a suitable constant depending only on A and k, and let $Q_1 = (\log X)^B$. We put

$$\mathfrak{S}_k(n, Q) = \sum_{q \leq Q} \frac{\mu(q)}{\varphi(q)} \prod_{p \mid q} (\rho_n(p) - 1),$$

where μ and φ denote the Möbius function and Euler's totient function, respectively. By standard application of the circle method, we have

$$R_k(n) = \mathfrak{S}_k(n, Q_1) \frac{n^{1/k}}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right) + \tilde{R}_k(n), \tag{2}$$

with

$$\sum_{n \leq X} |\tilde{R}_k(n)|^2 \ll X^{1+\frac{2}{k}} (\log X)^{-3A}. \tag{3}$$

Making use of (8) in Lemma 2, below, we obtain easily

$$\sum_{n \leq X} |\mathfrak{S}_k(n, \sqrt{X}) - \mathfrak{S}_k(n, Q_1)|^2 \ll X (\log X)^{-3A}. \tag{4}$$

In view of (2), (3) and (4), we have

$$R_k(n) = \mathfrak{S}_k(n, \sqrt{X}) \frac{n^{1/k}}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right) + O(X^{\frac{1}{k}} (\log X)^{-A}), \tag{5}$$

for $n \leq X$ with at most $O(X(\log X)^{-A})$ exceptions.

§3. In order to investigate $\mathfrak{S}_k(n, \sqrt{X})$, we define the function

$$Z_n(s) = \prod_p \left(1 - \frac{\rho_n(p) - 1}{(p - 1)p^{s-1}}\right),$$

where $s = \sigma + it$ is a complex variable, as usual. We write $b = 1 + (\log X)^{-1}$ and $T_0 = \exp(\sqrt{\log X})/2$, and apply Perron's formula (see [2, p. 105 Lemma], for example). Then we have routinely for $n \leq X$

$$\mathfrak{S}_k(n, \sqrt{X}) = \frac{1}{2\pi i} \int_{b-iT_0}^{b+iT_0} Z_n(s) \frac{\sqrt{X}^{s-1}}{s - 1} ds + O\left(\frac{(\log X)^k}{T_0}\right). \tag{6}$$

So we need some information about $Z_n(s)$ near the line $\sigma = 1$.

On the other hand, let $\zeta(s)$ and $\zeta_n(s)$ be the Riemann zeta function and the Dedekind zeta function of the field $\boldsymbol{Q}(n^{1/k})$, respectively, and let $N(n; \alpha, T)$ be the number of zeros of $\zeta_n(s)/\zeta(s)$ in region $\sigma \geq \alpha$ and $|t| \leq T$. Here we note that $\zeta_n(s)/\zeta(s)$ is an entire function (see [8]).

The Euler product for $\zeta_n(s)$ is written as

$$\zeta_n(s) = \prod_p \prod_{1 \leq f \leq k} (1 - p^{-fs})^{-a_n(f, p)},$$

with the number $a_n(f, p)$ of prime ideals $\mathfrak{p}$ in $\boldsymbol{Q}(n^{1/k})$ such that the norm of $\mathfrak{p}$ is p^f. In particular, we see

$$a_n(1, p) = \rho_n(p)$$

providing $n \in I_k$ and $p \nmid kn$. By observing the Euler product for $\zeta(s)/\zeta_n(s)$, we have

$$Z_n(s) = \frac{\zeta(s)}{\zeta_n(s)} \xi_n(s) \Xi_n(s),$$

where

$$\xi_n(s) = \prod_p \left\{(1-p^{-s})^{-\rho_n(p)+1}\left(1 - \frac{\rho_n(p)-1}{p^{s-1}(p-1)}\right)\right\} \prod_p \prod_{2 \le f \le k} (1-p^{-fs})^{-a_n(f,p)},$$

and

$$\Xi_n(s) = \prod_{p \mid kn} (1-p^{-s})^{\rho_n(p)-a_n(1,p)}.$$

It is quite easy to treat the functions $\xi_n(s)$ and $\Xi_n(s)$ near the line $\sigma = 1$. Thus we can regard $Z_n(s)$ as $\zeta(s)/\zeta_n(s)$ essentially.

Next, for a constant $0 < c < 1 - (\log(k-1))/(\log(k+1))$, we assume that $N(n; 1-c, 2T_0) = 0$, and put $\eta = c/32$. Then the function $Z_n(s)$ is analytic in the region $\sigma > 1-c$, $|t| \le 2T_0$, and Hadamard's three circle theorem gives

$$\max_{\substack{1-\eta \le \sigma \le 1+\eta \\ |t| \le T_0}} |Z_n(s)| \ll \exp(c_0(\log X)^{1/4}),$$

where $c_0 > 0$ is a constant. Therefore, on the integral in the right-hand side of (6), we see

$$\frac{1}{2\pi i}\int_{b-iT_0}^{b+iT_0} Z_n(s) \frac{\sqrt{X}^{\,s-1}}{s-1} ds = \mathfrak{S}_k(n) + \frac{1}{2\pi i}\left(\int_{b-iT_0}^{1-\eta-iT_0} + \int_{1-\eta-iT_0}^{1-\eta+iT_0} + \int_{1-\eta+iT_0}^{b+iT_0}\right)$$
$$= \mathfrak{S}_k(n) + O(T_0^{-1/2}).$$

Hence we have the following:

Lemma 1. *Assume that* $n \le X$, *and* $N(n; 1-c, \exp(\sqrt{\log X})) = 0$ *with some positive contsant* c. *Then we have*

$$\mathfrak{S}_k(n, \sqrt{X}) = \mathfrak{S}_k(n) + O\left(\exp\left(-\frac{1}{2}\sqrt{\log X}\right)\right).$$

We also obtain

$$\mathfrak{S}_k(n) = Z_n(1) \gg (\log 3n)^{-k}$$

by the known upper estimate for the residue of $\zeta_n(s)$ at $s = 1$. Therefore we conclude from (5) that the asymptotic formula (1) holds for $n \le X$ satisfying $n \in I_k$ and $N(n; 1-c, \exp(\sqrt{\log X})) = 0$ with some $c > 0$.

§4. It is easily seen that the number of the natural numbers $n \le X$ with $n \notin I_k$ is $O(\sqrt{X})$. So, it suffices for the proof of our Theorem to show that there exist positive constants c and δ, depending only on k, such that

(7) $$\sum_{\substack{n \le X \\ n \in I_k}} N(n; 1-c, \exp(\sqrt{\log X})) \ll X^{1-\delta}.$$

At this stage, we find the most important difference between $k = 2$ and $k \ge 3$. When $k = 2$, the function $\zeta_n(s)/\zeta(s)$ is the Dirichlet L function for a certain real primitive character, unless n is a square. Therefore Bombieri's zero density theorem for L functions is effectual for the treatment of the singular series $\mathfrak{S}_2(n, \sqrt{X})$ (see Miech [6]). For $k \ge 3$, however, we can not utilize such a known result. We shall prove a zero density theorem for $\zeta_n(s)/\zeta(s)$'s. To this end, we use the following Lemma 2.

Lemma 2. *We put* $\beta_n(m) = \mu(m)^2 \prod_{p|m}(\rho_n(p) - 1)$. *Let* ε *be any fixed positive constant, and let* $\{a_m\}$ *be any sequence of complex numbers. Then we have*

$$(8)\qquad \sum_{n \leq X} \Big| \sum_{M-N<m\leq M} a_m \beta_n(m) \Big|^2 \ll (X + NM) M^{\varepsilon} \sum_{M-N<m\leq M} |a_m|^2.$$

Further, assume that $M^{2(r+1)} \leq X^{r(r-1)}$ *for a fixed natural number* r. *Then we have*

$$\sum_{n\leq X} \Big| \sum_{m\leq M} a_m \beta_n(m) \Big|^2 \ll X^{1+\varepsilon} \sum_{m \leq M} |a_m|^2 + X^{1-\frac{1}{r+1}+\varepsilon} \max_{M_1 \leq M} \Big(M_1 \max_{M_1<m\leq 2M_1} |a_m| \Big)^2.$$

Through the same argument as in the study of the zero density of Dirichlet L functions, except that we employ the above Lemma 2 instead of the large sieve inequality, we obtain the following zero density estimate for $\zeta_n(s)/\zeta(s)$'s.

Lemma 3. *Let* $T \geq 1$, *and let* $\sigma_1 = 1 - (r(r-1))^{-1}$ *with a natural number* r. *Suppose that*

$$\sigma_1 > \frac{\log(k-1)}{\log(k+1)} \text{ and } (XT)^{(r+1)(k-1)(3-2\sigma_1)} \leq X^{r(r-1)}.$$

Then we have for $\sigma_1 \leq \sigma < 1$

$$\sum_{\substack{n\leq X\\ n\in I_k}} N(n;\sigma, T) \ll (XT)^{1-\frac{\sigma-\sigma_1}{3-\sigma-\sigma_1}+\varepsilon}$$

with any fixed $\varepsilon > 0$.

We apply Lemma 3 with $T = \exp(\sqrt{\log X})$, $r = k+1$, $\sigma_1 = 1 - (k(k+1))^{-1}$, then we have (7), as required, for $c = (2k(k+1))^{-1}$ and $\delta = (2k^2 + 2k + 4)^{-1}$ for example.

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