8. Bernstein-Gelfand-Gelfand Resolution for Generalized Kac-Moody Algebras

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Notation. Let $A = (a_{ij})_{i,j \in I}$ be a real $n \times n$ matrix satisfying the following conditions: (1) $a_{ii} = 2$, or $a_{ii} \leq 0$; (2) $a_{ij} \leq 0$ $(i \neq j)$, and $a_{ij} \in \boldsymbol{Z}$ if $a_{ii} = 2$; (3) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$. We call such a matrix a GGCM. Let $\mathfrak{g}(A)$ be a generalized Kac-Moody algebra (= GKM algebra), over the complex number field $\boldsymbol{C}$, with Cartan subalgebra $\mathfrak{h}$, the set of simple roots $\Pi = \{\alpha_i\}_{i \in I}$, and the set of simple coroots $\Pi^\vee = \{\alpha_i^\vee\}_{i \in I}$. Then, we have the root space decomposition: $\mathfrak{g}(A) = \mathfrak{h} \oplus \sum_{\alpha \in \Delta}^{\oplus} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space attached to a root $\alpha \in \Delta = \Delta^+ \cup \Delta^- \subset \mathfrak{h}^*$.

Let J be a finite type subset of the index set I, that is, a subset of I such that the submatrix $A_J := (a_{ij})_{i,j \in J}$ of $A = (a_{ij})_{i,j \in I}$ is a direct sum of generalized Cartan matrices of finite type. Corresponding to such a subset J of I, we define the following Lie subalgebras of $\mathfrak{g}(A)$ and subset of the Weyl group W:

$$\mathfrak{u}^\pm := \sum_{\alpha \in \Delta^+(J)}^{\oplus} \mathfrak{g}_{\pm\alpha},\ \mathfrak{m} := \mathfrak{h} \oplus \sum_{\alpha \in \Delta_J^+}^{\oplus} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),\ \mathfrak{p} := \mathfrak{m} \oplus \mathfrak{u}^+,$$

$$W(J) := \{w \in W \mid \Delta^+ \cap w(\Delta^-) \subset \Delta^+(J)\},$$

where $\Delta_J^+ := \Delta^+ \cap (\sum_{i \in J} \boldsymbol{Z}_{\geq 0} \alpha_i)$, $\Delta^+(J) := \Delta^+ \setminus \Delta_J^+$.

§1. The existence of the weak BGG resolution for GKM algebras. Let J be a finite type subset of I. We put $P^+ := \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha_i^\vee \rangle \geq 0\ (i \in I),\ \langle \mu, \alpha_i^\vee \rangle \in \boldsymbol{Z}_{\geq 0}$ if $a_{ii} = 2\}$, $P_J^+ := \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha_j^\vee \rangle \in \boldsymbol{Z}_{\geq 0}\ (j \in J)\}$. Let $L(\Lambda)\ (\Lambda \in P^+)$ be the irreducible highest weight $\mathfrak{g}(A)$-module with highest weight Λ, $L_\mathfrak{m}(\lambda)\ (\lambda \in P_J^+)$ the irreducible highest weight $\mathfrak{m}$-module with highest weight λ, and $V_\mathfrak{m}(\lambda) = U(\mathfrak{g}(A)) \otimes_{U(\mathfrak{p})} L_\mathfrak{m}(\lambda)\ (\lambda \in P_J^+)$ the generalized Verma module with highest weight λ. Note that the Verma modules $V(\lambda)\ (\lambda \in \mathfrak{h}^*)$ are precisely the generalized Verma modules $V_\mathfrak{h}(\lambda)$ for the case $J = \emptyset$.

From now on throughout this paper, we assume that the GGCM $A = (a_{ij})_{i,j \in I}$ is symmetrizable. Then, there exists a positive diagonal matrix D such that $D^{-1}A$ is symmetric.

Let $\Pi^{re} := \{\alpha_i \in \Pi \mid a_{ii} = 2\}$ be the set of real simple roots, and $\Pi^{im} := \{\alpha_i \in \Pi \mid a_{ii} \leq 0\}$ the set of imaginary simple roots of $\mathfrak{g}(A)$. For $\Lambda \in P^+$, we denote by $\mathcal{S}(\Lambda)$ the set of sums of distinct, pairwise perpendicular, imaginary simple roots $\alpha_i \in \Pi^{im}$ with $\langle \Lambda, \alpha_i^\vee \rangle = 0$. Here, for α_i, $\alpha_j \in \Pi^{im} (i \neq j)$, α_i and α_j are said to be perpendicular if $a_{ij} = a_{ji} = 0$. We simply write $\mathcal{S}$ for $\mathcal{S}(0)$, $0 \in \mathfrak{h}^*$.

We have the following lemma for the relative Ext bifunctor $\mathrm{Ext}^1_{(\mathfrak{g}(A),\mathfrak{m})}$

(cf. [6]), defined in the category $\mathscr{C}(\mathfrak{g}(A), \mathfrak{m})$ of all $\mathfrak{g}(A)$-modules which decompose into direct sums of finite dimensional irreducible $\mathfrak{m}$-modules.

Lemma 1.1. *Let $\Lambda \in P^+$, $w_i \in W(J)$, and $\beta_i \in \mathscr{S}(\Lambda)$ $(i = 1, 2)$.* If

$$\mathrm{Ext}^1_{(\mathfrak{g}(A),\mathfrak{m})}(V_{\mathfrak{m}}(w_1(\Lambda + \rho - \beta_1) - \rho), V_{\mathfrak{m}}(w_2(\Lambda + \rho - \beta_2) - \rho)) \neq 0,$$

then we have $\ell(w_1) + ht(\beta_1) \lneqq \ell(w_2) + ht(\beta_2)$.
Here, ρ is a fixed element of $\mathfrak{h}^*$ such that $\langle \rho, \alpha_i^\vee \rangle = (1/2) \cdot a_{ii}$ $(i \in I)$, $\ell(w)$ $(w \in W)$ is the length of w, and for $\beta = \sum_{i \in I} k_i \alpha_i$ $(k_i \in \boldsymbol{Z}_{\geq 0})$, we put $ht(\beta) = \sum_{i \in I} k_i$.

From now on, we write $(w, \beta) \circ \Lambda = w(\Lambda + \rho - \beta) - \rho$ for $(w, \beta) \in W \times \mathscr{S}(\Lambda)$.

By the arguments similar to the ones in [3], [7], and [8], using Lemma 1.1, we can prove the following theorem, which generalizes a classical result of Bernstein-Gelfand-Gelfand (= BGG) to GKM algebras (cf. [1]).

Theorem 1.2 (Existence of the weak BGG resolution). *Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable* GGCM. *Then, for the irreducible highest weight module $L(\Lambda)$ with highest weight $\Lambda \in P^+$ over the* GKM *algebra $\mathfrak{g}(A)$, there exists a $\mathfrak{g}(A)$-module exact sequence*:

$$0 \leftarrow L(\Lambda) \overset{\partial_0}{\leftarrow} C_0(\Lambda) \overset{\partial_1}{\leftarrow} C_1(\Lambda) \overset{\partial_2}{\leftarrow} C_2(\Lambda) \overset{\partial_3}{\leftarrow} \cdots,$$

$$\text{where } C_p(\Lambda) = \sum_{\substack{w \in W(J), \beta \in \mathscr{S}(\Lambda) \\ \ell(w) + ht(\beta) = p}}^{\oplus} V_{\mathfrak{m}}((w, \beta) \circ \Lambda) \ (p \geq 0).$$

§2. Homology vanishing theorems.

Here, as before, we assume that J is a finite type subset of I. From Theorem 1.2, we obtain the following extension of Kostant's homology theorem to GKM algebras.

Proposition 2.1. *Let $\Lambda \in P^+$. Then, as $\mathfrak{m}$-modules,*

$$H_p(\mathfrak{u}^-, L(\Lambda)) \cong \sum_{\substack{w \in W(J), \beta \in \mathscr{S}(\Lambda) \\ \ell(w) + ht(\beta) = p}}^{\oplus} L_{\mathfrak{m}}((w, \beta) \circ \Lambda) \ (p \geq 0).$$

Here, the sum is a direct sum of inequivalent irreducible (highest weight) $\mathfrak{m}$-modules.

From Proposition 2.1, we can derive the following proposition on Lie algebra homology by the same argument as in [5]. For the notation, see [6].

Proposition 2.2. *Let $\Lambda \in P^+$, $\mu \in P_J^+$. If $\mu \neq (w, \beta) \circ \Lambda$ for any $w \in W(J)$, $\beta \in \mathscr{S}(\Lambda)$, then*

$$\mathrm{Tor}_n^{\mathfrak{g}(A)} (L^*(\Lambda), V_{\mathfrak{m}}(\mu)) = 0 \ (n \geq 0),$$

$$\mathrm{Tor}_n^{(\mathfrak{g}(A),\mathfrak{m})} (L^*(\Lambda), V_{\mathfrak{m}}(\mu)) = 0 \ (n \geq 0).$$

Here, $L^*(\Lambda)$ is the irreducible lowest weight $\mathfrak{g}(A)$-module with lowest weight $-\Lambda$.

By Theorem 1.2 and Proposition 2.2, we get the following theorem.

Theorem 2.3. *Let $\Lambda_1, \Lambda_2 \in P^+$. Assume that $\Lambda_1 - \Lambda_2 \neq \beta_1 - \beta_2$ for any $\beta_i \in \mathscr{S}(\Lambda_i)$ $(i = 1, 2)$. Then, we have*

$$\mathrm{Tor}_n^{\mathfrak{g}(A)} (L^*(\Lambda_1), L(\Lambda_2)) = 0 \ (n \geq 0),$$

$$\mathrm{Tor}_n^{(\mathfrak{g}(A),\mathfrak{m})} (L^*(\Lambda_1), L(\Lambda_2)) = 0 \ (n \geq 0).$$

Corollary 2.4. *Let $\Lambda \in P^+$. Assume that $\Lambda \neq \beta_1 - \beta_2$ for any $\beta_1 \in \mathscr{S}(\Lambda)$, $\beta_2 \in \mathscr{S}$. Then,*

$$H_n(\mathfrak{g}(A), L(\Lambda)) = 0 \ (n \geq 0),$$
$$H_n(\mathfrak{g}(A), \mathfrak{m}, L(\Lambda)) = 0 \ (n \geq 0).$$

For the relative Lie algebra homology $H_n(\mathfrak{g}(A), \mathfrak{m}, \mathbf{C})$ $(n \geq 0)$ with $\mathbf{C}$ the trivial one dimensional $\mathfrak{g}(A)$-module, we have the following, as a generalization of [6, Corollary 6.7].

Theorem 2.5. $H_{2s+1}(\mathfrak{g}(A), \mathfrak{m}, \mathbf{C}) = 0$ $(s \geq 0)$, *and*

$\dim_{\mathbf{C}} H_{2s}(\mathfrak{g}(A), \mathfrak{m}, \mathbf{C})$ $(s \geq 0)$

$=$ *the number of elements of the set* $\{(w, \beta) \in W(J) \times \mathcal{S} \mid \ell(w) + ht(\beta) = s\}$

$=$ *the number of* $\mathfrak{m}$-*irreducible components in the Lie algebra homology* $H_s(\mathfrak{u}^-, \mathbf{C})$.

§3. Verma module embeddings. Let $\Delta^{re} := W \cdot \Pi^{re}$ be the set of real roots, and $\Delta^{im} := \Delta \setminus \Delta^{re}$ the set of imaginary roots. For $\alpha \in \Delta^{re}$, we define a reflection r_α with respect to α by $r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ $(\lambda \in \mathfrak{h}^*)$, where $\alpha^\vee$ is the dual real root of α.

Definition (Bruhat ordering). Let $w_1, w_2 \in W$. We write $w_1 \leftarrow w_2$ if there exists some $\gamma \in \Delta^{re} \cap \Delta^+$ such that $w_1 = r_\gamma w_2$ and $\ell(w_1) = \ell(w_2) + 1$. Moreover, for $w, w' \in W$, we write $w \leqslant w'$ if $w = w'$ or if there exist $w_1, \ldots, w_r \in W$ such that

$$w = w_0 \leftarrow w_1 \leftarrow \cdots \leftarrow w_r \leftarrow w_{r+1} = w'.$$

Definition. Let $\beta_1, \beta_2 \in \mathcal{S}$. We write $\beta_1 \leftarrow \beta_2$ if there exists some $\alpha_j \in \Pi^{im}$ such that $\beta_1 = \beta_2 + \alpha_j$. Moreover, for $\beta = \sum_{k \in K} \alpha_k$, $\beta' = \sum_{l \in L} \alpha_l \in \mathcal{S}$, we write $\beta \geqslant \beta'$ if $K \supset L$.

Definition. For $(w_1, \beta_1), (w_2, \beta_2) \in W \times \mathcal{S}$, we write $(w_1, \beta_1) \leftarrow (w_2, \beta_2)$ if $w_1 \leftarrow w_2$ and $\beta_1 = \beta_2$, or if $w_1 = w_2$ and $\beta_1 \leftarrow \beta_2$.

Remark. Let $(w_1, \beta_1), (w_2, \beta_2) \in W \times \mathcal{S}$. Then, the number of elements $(w, \beta) \in W \times \mathcal{S}$ such that $(w_1, \beta_1) \leftarrow (w, \beta) \leftarrow (w_2, \beta_2)$ is 0 or 2.

We can prove the following generalization of one of Verma's classical results, using the theory of Enright's completion functors.

Proposition 3.1. *Fix* $\Lambda \in P^+$. *Let* $(w_1, \beta_1), (w_2, \beta_2) \in W \times \mathcal{S}(\Lambda)$. *Then, we have*

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathfrak{g}(A)}(V((w_1, \beta_1) \circ \Lambda), V((w_2, \beta_2) \circ \Lambda)) \leq 1.$$

In the case where the equality holds in Proposition 3.1, we write

$$V((w_1, \beta_1) \circ \Lambda) \subset V((w_2, \beta_2) \circ \Lambda).$$

Proposition 3.2. *Let* $\Lambda \in P^+$, $(w_1, \beta_1), (w_2, \beta_2) \in W \times \mathcal{S}(\Lambda)$. *Then,*

$$V((w_1, \beta_1) \circ \Lambda) \subset V((w_2, \beta_2) \circ \Lambda)$$
$$\Leftrightarrow \ w_1 \leqslant w_2, \ \beta_1 \geqslant \beta_2$$
$$\Leftrightarrow \ [V((w_2, \beta_2) \circ \Lambda) : L((w_1, \beta_1) \circ \Lambda)] \neq 0.$$

Here, for $\lambda, \mu \in \mathfrak{h}^*$, $[V(\lambda) : L(\mu)]$ *denotes the multiplicity of* $L(\mu)$ *in* $V(\lambda)$.

Now, let J be a finite type subset of I, and $\lambda \in P_J^+$. The generalized Verma module $V_{\mathfrak{m}}(\lambda)$ with highest weight λ is a quotient of the Verma module $V(\lambda)$ with highest weight λ. We denote by $K(\lambda)$ the kernel of the natural quotient map of $V(\lambda)$ onto $V_{\mathfrak{m}}(\lambda)$. If for $\lambda, \mu \in P_J^+$, $f : V(\lambda) \to V(\mu)$ is a nonzero $\mathfrak{g}(A)$-module map, then we can easily see that $f(K(\lambda)) \subset K(\mu)$ by a classical result of Harish-Chandra. Therefore, f naturally determines a $\mathfrak{g}(A)$-module map $\hat{f}: V_{\mathrm{m}}(\lambda) \to V_{\mathrm{m}}(\mu)$ such that $\hat{f}(v + K(\lambda)) = f(v) + K(\mu)\,(v \in V(\lambda))$. We call this map the *standard map* associated to f.

Then, by Proposition 3.2, we can prove the following.

Proposition 3.3. *Let $\Lambda \in P^+$, and let $(w_1, \beta_1), (w_2, \beta_2) \in W(J) \times \mathcal{B}(\Lambda)$ be such that $\ell(w_1) + ht(\beta_1) = \ell(w_2) + ht(\beta_2) + 1$. Then, there exists a nonzero $\mathfrak{g}(A)$-module map $V_{\mathrm{m}}((w_1, \beta_1) \circ \Lambda) \to V_{\mathrm{m}}((w_2, \beta_2) \circ \Lambda)$ if and only if $(w_1, \beta_1) \leftarrow (w_2, \beta_2)$. In this case, the standard map associated to the inclusion of $V((w_1, \beta_1) \circ \Lambda)$ into $V((w_2, \beta_2) \circ \Lambda)$ is also nonzero.*

§4. Construction of the strong BGG resolution. Theorem 1.2 gives no informations about the $\mathfrak{g}(A)$-module maps $\partial_p\,(p \geq 0)$. Here, we give an explicit construction of the strong BGG resolution, which is equivalent to the weak BGG resolution in Theorem 1.2.

Definition. Let us call a quadruple $\{(w_1, \beta_1), (w_2, \beta_2), (w_3, \beta_3), (w_4, \beta_4)\}$ of elements of $W \times \mathcal{B}$ a *square* if

$$(w_1, \beta_1) \leftarrow (w_i, \beta_i) \leftarrow (w_4, \beta_4)\ (i = 2, 3), \quad (w_2, \beta_2) \neq (w_3, \beta_3).$$

Lemma 4.1. *To each arrow $(w_1, \beta_1) \leftarrow (w_2, \beta_2)$, we can associate a number $c((w_1, \beta_1), (w_2, \beta_2)) \in \{1, -1\}$ such that the product of all numbers associated to the four arrows of any square $\{(w_1, \beta_1), (w_2, \beta_2), (w_3, \beta_3), (w_4, \beta_4)\}$ is equal to -1.*

Let $\Lambda \in P^+$. Then, by Propositions 3.1 and 3.2, we can fix an injection $\iota_{(w_1,\beta_1),(w_2,\beta_2)}: V((w_1, \beta_1) \circ \Lambda) \to V((w_2, \beta_2) \circ \Lambda)$, for each pair (w_1, β_1), $(w_2, \beta_2) \in W \times \mathcal{B}(\Lambda)$ in such a way that $\iota_{(w_2,\beta_2),(1,0)} \circ \iota_{(w_1,\beta_1),(w_2,\beta_2)} = \iota_{(w_1,\beta_1),(1,0)}$.

Now, we temporarily assume that $J = \emptyset$. Remark that, in this case,

$$C_p(\Lambda) = \sum^{\oplus}_{\substack{w \in W, \beta \in \mathcal{B}(\Lambda) \\ \ell(w) + ht(\beta) = p}} V((w, \beta) \circ \Lambda)\ (p \geq 0)$$

in the weak BGG resolution in Theorem 1.2. The next theorem gives an explicit construction of the strong BGG resolution.

Theorem 4.2. *Let $\Lambda \in P^+$. For each $p \in \mathbf{Z}_{\geq 1}$, let $d_p: C_p(\Lambda) \to C_{p-1}(\Lambda)$ be the map defined by*

$$d_p := \bigoplus_{\substack{\ell(w_1) + ht(\beta_1) = p \\ \ell(w_2) + ht(\beta_2) = p-1}} d^p_{(w_1,\beta_1),(w_2,\beta_2)} \cdot \iota_{(w_1,\beta_1),(w_2,\beta_2)},$$

$$\text{where } d^p_{(w_1,\beta_1),(w_2,\beta_2)} := \begin{cases} c((w_1, \beta_1), (w_2, \beta_2)) & \text{if } (w_1, \beta_1) \leftarrow (w_2, \beta_2) \\ 0 & \text{otherwise,} \end{cases}$$

and let $d_0: C_0(\Lambda) = V(\Lambda) \to L(\Lambda)$ be a canonical surjection. Then, we have the following $\mathfrak{g}(A)$-module exact sequence, which is equivalent to the weak BGG *resolution in Theorem* 1.2 *for the case $J = \emptyset$:*

$$0 \leftarrow L(\Lambda) \xleftarrow{d_0} C_0(\Lambda) \xleftarrow{d_1} C_1(\Lambda) \xleftarrow{d_2} C_2(\Lambda) \xleftarrow{d_3} \cdots,$$
$$\text{where } C_p(\Lambda) = \sum^{\oplus}_{\substack{w \in W, \beta \in \mathcal{B}(\Lambda) \\ \ell(w) + ht(\beta) = p}} V((w, \beta) \circ \Lambda)\ (p \geq 0).$$

We now return to the case where J is an arbitrary finite type subset of I. Let $\Lambda \in P^+$. Note that $(w, \beta) \circ \Lambda \in P_J^+$ for $(w, \beta) \in W(J) \times \mathcal{B}(\Lambda)$. For each $p \in \mathbf{Z}_{\geq 1}$, let $\hat{d}_p: C_p(\Lambda) \to C_{p-1}(\Lambda)$ be the map defined by

$$\hat{d}_p := \bigoplus_{\substack{\ell(w_1)+ht(\beta_1)=p \\ \ell(w_2)+ht(\beta_2)=p-1}} d^p_{(w_1,\beta_1),(w_2,\beta_2)} \cdot \hat{\iota}_{(w_1,\beta_1),(w_2,\beta_2)},$$

where, for $(w_1, \beta_1), (w_2, \beta_2) \in W(J) \times \mathcal{S}(\Lambda)$, $\hat{\iota}_{(w_1,\beta_1),(w_2,\beta_2)} : V_{\mathrm{m}}((w_1, \beta_1) \circ \Lambda) \to V_{\mathrm{m}}((w_2, \beta_2) \circ \Lambda)$ is the standard map associated to the inclusion $\iota_{(w_1,\beta_1),(w_2,\beta_2)} : V((w_1, \beta_1) \circ \Lambda) \to V((w_2, \beta_2) \circ \Lambda)$, and the number $d^p_{(w_1,\beta_1),(w_2,\beta_2)}$ is as in Theorem 4.2, restricted to $W(J) \times \mathcal{S}(\Lambda)$. By the classical result of Harish-Chandra, we can easily see that there exists a surjective $\mathfrak{g}(A)$-module map $\hat{\eta} : V_{\mathrm{m}}(\Lambda) \to L(\Lambda)$, which takes a highest weight vector generating $V_{\mathrm{m}}(\Lambda)$ to a highest weight vector of $L(\Lambda)$. Then, we can prove the following extension of Theorem 4.2 by exactly the same argument as the one for [7, Theorem 11.4] or [8, Theorem 9.12].

Theorem 4.3. *Let $\Lambda \in P^+$, and J be an arbitrary finite type subset of I. Then, we have the following $\mathfrak{g}(A)$-module exact sequence, which is equivalent to the weak* BGG *resolution in Theorem* 1.2 :

$$0 \leftarrow L(\Lambda) \overset{\hat{\eta}}{\leftarrow} C_0(\Lambda) \overset{\hat{d}_1}{\leftarrow} C_1(\Lambda) \overset{\hat{d}_2}{\leftarrow} C_2(\Lambda) \overset{\hat{d}_3}{\leftarrow} \cdots,$$

where $C_p(\Lambda) = \sum^{\oplus}_{\substack{w\in W(J),\beta\in\mathcal{S}(\Lambda) \\ \ell(w)+ht(\beta)=p}} V_{\mathrm{m}}((w, \beta) \circ \Lambda) \ (p \geq 0).$

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