58. *Normal Band Compositions of Semigroups*[*)]

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Abstract: In this paper we give a construction of bands of arbitrary semigroups and we apply this result to study of normal bands of semigroups, and especially for normal bands of monoids. We generalize some well-known results concerning normal bands of monoids and groups.

In this paper we consider band compositions in the general case. Using a general construction for a semilattice of semigroups, we give a construction for a band of arbitrary semigroups. This construction is a very simple consequence of Theorem A, but we give some important applications of this construction: We give a description of normal bands of arbitrary semigroups, especially of normal bands of monoids, and as consequences we obtain some well-known results concerning normal bands of monoids and groups. Note that in our considerations, the conditions (5) and (6) in Theorem A have the important role.

Throughout this paper, $S=(B\,;S_i)$ means that a semigroup S is a band B of semigroups S_i, $i\in B$. Let $S=(B\,;S_i)$, where each S_i is a monoid with the identity e_i, S is a *systematic band* B of S_i, $i\in B$, if $ij=j\Rightarrow e_ie_j=e_j$ and $ji=j\Rightarrow e_je_i=e_j$ (M. Yamada [14]). S is a *proper band* of S_i if $\{e_i\mid i\in B\}$ is a subsemigroup of S (B.M. Schein [11]). Let S be an ideal of a semigroup D. A congruence σ on D is an *S-congruence* on D if its restriction on S is the equality relation on S. An ideal extension D of a semigroup S is a *dense extension* of S if the equality relation is the unique S-congruence on D.

Theorem A [9]. *Let Y be a semilattice. For each $\alpha\in Y$ we associate a semigroup S_α and an extension D_α of S_α such that $D_\alpha\cap D_\beta=\emptyset$ if $\alpha\neq\beta$. For every pair $\alpha,\beta\in Y$ such that $\alpha\geq\beta$ let $\phi_{\alpha,\beta}:S_\alpha\rightarrow D_\beta$ be a mapping satisfying:*

(1) *$\phi_{\alpha,\alpha}$ is the identity mapping on S_α;*

(2) $(S_\alpha\phi_{\alpha,\alpha\beta})(S_\beta\phi_{\beta,\alpha\beta})\subseteq S_{\alpha\beta}$;

(3) $[(a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta})]\phi_{\alpha\beta,\gamma}=(a\phi_{\alpha,\gamma})(b\phi_{\beta,\gamma})$,

for all $\alpha,\beta,\gamma\in Y$ such that $\alpha\beta>\gamma$ and all $a\in S_\alpha$, $b\in S_\beta$.
Define a multiplication $*$ on $S=\cup_{\alpha\in Y}S_\alpha$ with:

(4) $$a*b=(a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}),\quad(a\in S_\alpha,\ b\in S_\beta).$$

Then S is a semilattice Y of semigroups S_α, in notation $S=(Y\,;S_\alpha,\phi_{\alpha,\beta},D_\alpha)$.

Conversely, every semigroup S which is a semilattice Y of semigroups S_α can be so constructed. In addition, D_α can be chosen to satisfy:

(5) $D_\alpha=\{b\phi_{\beta,\alpha}\mid\beta\geq\alpha,\ b\in S_\beta,\ \beta\in Y\}$;

(6) *D_α is a dense extension of S_α.*

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If we assume $\alpha=\beta$ in (3), then $\phi_{\alpha,\gamma}$ is a homomorphism for all $\alpha, \gamma \in Y$ such that $\alpha \geq \gamma$. If each $\phi_{\alpha,\beta}$ maps S_α into S_β, i.e. if $S_\alpha\phi_{\alpha,\beta} \subseteq S_\beta$, or if $D_\alpha = S_\alpha$, for each $\alpha \in Y$, then we write $S=(Y\,;S_\alpha,\ \phi_{\alpha,\beta})$. In this case the condition (2) can be omitted. If $S=(Y\,;S_\alpha,\ \phi_{\alpha,\beta})$ and if $\{\phi_{\alpha,\beta} \mid \alpha \geq \beta\}$ is a *transitive system of homomorphisms*, i.e. if $\phi_{\alpha,\beta}\phi_{\beta,\gamma}=\phi_{\alpha,\gamma}$, for $\alpha \geq \beta \geq \gamma$, then S is a *strong semilattice* of semigroups S_α and we write $S=[Y\,;S_\alpha,\ \phi_{\alpha,\beta}]$.

For undefined notions and notations we refer to [9] and [10].

A very important problem in the theory of semigroups is the following: Given a family $\{S_i \mid i \in B\}$ of semigroups indexed by a band B, how to define a multiplication on $S=\cup_{i\in B} S_i$ such that $S=(B\,;S_i)$, i.e. such that $S_iS_j \subseteq S_{ij}$, for all $i, j \in B$? In such a case, we say that S is a *band composition of semigroups* S_i. Band compositions have been considered merely in various special cases. Left, right and matrix compositions of semigroups were studied by R. Yoshida [15], [16], M. Petrich [10] and Š. Schwarz [12]. A composition of a semilattice of arbitrary semigroups is given by Theorem A ([9]). Some special types of such compositions were studied by G. Lallement [6] and M. Petrich [8]. Strong semilattices of semigroups were first defined and studied by A.H. Clifford [5], and then by M. Yamada and N. Kimura [13], M. Petrich [7], M. Yamada [14]. Some band compositions were considered by the authors [2], and compositions of bands of monoids were considered by B.M. Schein [11], M. Yamada [14] and by the authors [1], [3]. Band compositions obtained from spined products of some semigroups will be presented in the next paper of the authors [4].

Theorem 1. *Let a band B be a semilattice Y of rectangular bands B_α, $\alpha \in Y$. To each $i \in B$ we associate a semigroup S_i such that $S_i \cap S_j = \emptyset$ if $i \neq j$. Then a semigroup S is a band B of semigroups S_i, $i \in B$, if and only if the following conditions hold*:

(7) $S=(Y\,;S_\alpha,\ \phi_{\alpha,\beta},\ D_\alpha)$;

(8) *each S_α is a matrix B_α of semigroups S_i, $i \in B_\alpha$, and D_α is an ideal extension of S_α* ;

(9) $(S_i\phi_{\alpha,\alpha\beta})(S_j\phi_{\beta,\alpha\beta}) \subseteq S_{ij}$, *for all $i \in B_\alpha$, $j \in B_\beta$.*

Proof. Let S be a band B of semigroups S_i, $i \in B$. Then S is a semilattice Y of semigroups S_α and for every $\alpha \in Y$, S_α is a matrix B_α of semigroups S_i, $i \in B_\alpha$. By Theorem A, we see (7) and (8), and by (4) we obtain (9).

Conversely, let (7), (8) and (9) hold. Then by (9) and by the definition of multiplication in B we obtain that S is a band B of semigroups S_i, $i \in B$.

A band B is *normal* if it is a strong semilattice of rectangular bands, or, equivalently, if it satisfies the identity $axya=ayxa$ ([10]).

Theorem 2. *Let S be a semigroup constructed as in Theorem* 1. *Then each D_α can be chosen to satisfy*:

($A1$) *D_α is a matrix B_α of semigroups D_i, $i \in B_\alpha$* ;

($A2$) *each S_i is contained in D_i*;

if and only if B is a normal band.

Proof. Let S be a band composition constructed as in Theorem 1 and let π be the related band congruence.

Assume $a, x, y \in S$, $a \in S_\alpha$, $x \in S_\beta$, $y \in S_\gamma$, $\alpha, \beta, \gamma \in Y$. Let $\delta = \alpha\beta\gamma$, and let $a\phi_{\alpha,\delta} \in D_i$, $x \in \phi_\beta$, $\delta \in D_j$, $y\phi_{\gamma,\delta} \in D_k$, $i, j, k \in B_\delta$. By (4), (3) and by (A1) we obtain

(10) $a * x * y * a = (a\phi_{\alpha,\beta})(x\phi_{\beta,\delta})(y\phi_{\gamma,\delta})(a\phi_{\alpha,\delta}) \in D_iD_jD_kD_i \subseteq D_i,$

(11) $a * y * x * a = (a\phi_{\alpha,\delta})(y\phi_{\gamma,\delta})(x\phi_{\beta,\delta})(a\phi_{\alpha,\delta}) \in D_iD_kD_jD_i \subseteq D_i.$

Thus, by (10) and (11) we have that $a * x * y * a$, $a * y * x * a \in D_i \cap S = S_i$, so $a * x * y * a \pi a * y * x * a$, whence $B \cong S/\pi$ is a normal band.

Conversely, let $B = [Y; B_\alpha, \theta_{\alpha,\beta}]$ be a normal band and each D_α be chosen to satisfy (5). Let $a \in S_i$, $b \in S_j$, where $i \in B_\alpha$, $j \in B_\beta$, $\alpha, \beta, \gamma \in Y$, $\alpha, \beta \geq \gamma$. Then

(12) $$a\phi_{\alpha,\gamma} = b\phi_{\beta,\gamma} \Rightarrow i\theta_{\alpha,\gamma} = j\theta_{\beta,\gamma}.$$

Indeed, let $a\phi_{\alpha,\gamma} = b\phi_{\beta,\gamma}$ and let $x \in S_{i\theta_{\alpha,\gamma}}$. Then $a * x \in S_i * S_{i\theta_{\alpha,\gamma}} \subseteq S_{i(i\theta_{\alpha,\gamma})}$, $b * x \in S_j * S_{i\theta_{\alpha,\gamma}} \subseteq S_{i(i\theta_{\alpha,\gamma})}$, so by $a * x = (a\phi_{\alpha,\gamma})x = (b\phi_{\beta,\gamma})x = b * x$ we obtain that $(j\theta_{\beta,\gamma})(i\theta_{\alpha,\gamma}) = j(i\theta_{\alpha,\gamma}) = i\theta_{\alpha,\gamma}$. Similarly we obtain that $(i\theta_{\alpha,\gamma})(j\theta_{\beta,\gamma}) = i\theta_{\alpha,\gamma}$, so $i\theta_{\alpha,\gamma} = j\theta_{\beta,\gamma}$. Thus, (12) holds. Assume that

$$D_k = \{a\phi_{\alpha,\gamma} \mid \alpha \geq \gamma,\ a \in S_i,\ i \in B_\alpha,\ i\theta_{\alpha,\gamma} = k\}, \quad \gamma \in Y,\ k \in B_\gamma.$$

By (12) it follows that these sets are pairwise disjoint. It is clear that $D_\gamma = \cup \{D_k \mid k \in B_\gamma\}$, $S_k \subseteq D_k$, for all $k \in B_\gamma$ and $S_i\phi_{\alpha,\gamma} \subseteq D_{i\theta_{\alpha,\gamma}}$, for all $\alpha \geq \gamma$, $i \in B_\alpha$.

Assume $a \in S_i$, $b \in S_j$, $\alpha, \beta \geq \gamma$, $\alpha, \beta, \gamma \in Y$, $i \in B_\alpha$, $j \in B_\beta$. Then $a * b \in S_{ij}$, so by (3) it follows that

$(a\phi_{\alpha,\gamma})(b\phi_{\beta,\gamma}) = ((a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}))\phi_{\alpha\beta,\gamma} = (a * b)\phi_{\alpha\beta,\gamma} \in S_{ij}\phi_{\alpha\beta,\gamma} \subseteq D_{(ij)\theta_{\alpha\beta,\gamma}}$.

Since $(ij)\theta_{\alpha\beta,\gamma} = ((i\theta_{\alpha,\alpha\beta})(j\theta_{\beta,\alpha\beta}))\theta_{\alpha\beta,\gamma} = (i\theta_{\alpha,\gamma})(j\theta_{\beta,\gamma})$, then $D_{i\theta_{\alpha,\gamma}}D_{j\theta_{\beta,\gamma}} \subseteq D_{(i\theta_{\alpha,\gamma})(j\theta_{\beta,\gamma})}$, so each D_γ is a matrix B_γ of semigroups D_k, $k \in B_\gamma$.

It is known that if S is a semilattice Y of monoids S_α, then this semilattice is composed as $(Y; S_\alpha, \phi_{\alpha,\beta})$ (since monoids have not proper dense extensions). This result can be generalized in the following way.

Theorem 3. *A semigroup S is normal band of monoids if and only if $S = (Y; S_\alpha, \phi_{\alpha,\beta})$ such that each S_α is a matrix of monoids.*

Proof. Let $B = [Y; B_\alpha, \theta_{\alpha,\beta}]$, where B_α are rectangular bands and let S be a normal band B of monoids S_i, $i \in B_\alpha$, $\alpha \in Y$, constructed as in Theorem 1, with (5) and (6), and let (A1) and (A2) from Theorem 2 hold. For $i \in B_\alpha$, let e_i be the identity element of S_i.

Assume $\alpha \in Y$. Define a relation σ on D_α by

$$a \sigma b \Leftrightarrow a, b \in D_i,\ i \in B_\alpha, \text{ and } ae_i = be_i.$$

It is clear that σ is an equivalence relation. Let $a \sigma b$ and $x \in D_j$. Note, firstly, that $ae_i = e_i(ae_i) = (e_ia)e_i = e_ia$, for all $a \in D_i$ since $e_ia, ae_i \in S_i$. Assume that $a, b \in D_i$ for some $i \in B_\alpha$. Then $ax, bx \in D_{ij}$, whence

$$(ax)e_{ij} = e_{ij}(ax) = (e_{ij}a)e_ix = e_{ij}(ae_i)x = e_{ij}(be_i)x = (bx)e_{ij},$$

so σ is a right congruence. Similarly we prove that σ is a left congruence, so σ is a congruence.

Let $a, b \in S_\alpha$ and let $a \sigma b$. Then $a, b \in S_i$, for some $i \in B_\alpha$, whence $a = ae_i = be_i = b$. Thus, σ is a S_α-congruence. Since D_α is a dense extension of S_α (by the hypothesis (6)), then σ is the equality relation on D_α.

Assume $a \in D_i$, for some $i \in B_\alpha$. Then by $a\,\sigma\, ae_i$ it follows that $a = ae_i \in S_i$. Therefore $D_\alpha = S_\alpha$.

Conversely, let $S = (Y\,; S_\alpha, \phi_{\alpha,\beta})$ and let each S_α be a matrix B_α of monoids S_i, $i \in B_\alpha$. Assume $\alpha, \beta \in Y$ such that $\alpha \geq \beta$. Let us prove that

(13) $$(\forall i \in B_\alpha)(\exists_1 j \in B_\beta)\ S_i\phi_{\alpha,\beta} \subseteq S_j.$$

Assume $i \in B_\alpha$ and assume that e is an identity element of S_i. Let $j \in B_\beta$ such that $e\phi_{\alpha,\beta} \in S_j$. Then for every $a \in S_i$ we obtain that

$$a\phi_{\alpha,\beta} = (eae)\phi_{\alpha,\beta} = (e\phi_{\alpha,\beta})(a\phi_{\alpha,\beta})(e\phi_{\alpha,\beta}) \in S_jS_\beta S_j \subseteq S_j.$$

Thus, $S_i\phi_{\alpha,\beta} \subseteq S_j$, and since S_k are pairwise disjoint, then (13) holds. Therefore, the mapping $\theta_{\alpha,\beta}$ of B_α into B_β given by:

$$i\theta_{\alpha,\beta} = j \Leftrightarrow S_i\phi_{\alpha,\beta} \subseteq S_j,$$

is well defined. It is not hard to verify that $\{\theta_{\alpha,\beta} \mid \alpha \geq \beta,\ \alpha, \beta \in Y\}$ constitutes a transitive system. If $B = [Y\,; \beta_\alpha, \theta_{\alpha,\beta}]$, then B is a normal band and for $a \in S_i$, $b \in S_j$, we have that

$$\begin{aligned} ab = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}) &\in (S_i\phi_{\alpha,\alpha\beta})(S_j\phi_{\beta,\alpha\beta}) \subseteq S_{i\theta_{\alpha,\alpha\beta}}S_{j\theta_{\beta,\alpha\beta}} \\ &\subseteq S_{(i\theta_{\alpha,\alpha\beta})(i\theta_{\beta,\alpha\beta})} = S_{ij}, \end{aligned}$$

so S is a band B of monoids S_i ($\alpha \in Y$, $i \in B_\alpha$).

Remark 1. Let $S = (B\,; S_i)$, B be a normal band and each S_i be a monoid with the identity e_i, and let $B = [Y\,; B_\alpha, \theta_{\alpha,\beta}]$, Y be a semilattice, B_α rectangular bands. By Theorem 3 it follows that this is equivalent to $S = (Y\,; S_\alpha, \phi_{\alpha,\beta})$, where $S_\alpha = (B_\alpha\,; S_i)$. Moreover, it can be proved that *each* $\phi_{\alpha,\beta}$, $\alpha, \beta \in Y$, $\alpha \geq \beta$, *is uniquely determined with* $a\phi_{\alpha,\beta} = ae_{i\theta_{\alpha,\beta}}$, *for* $a \in S_i$, $i \in B_\alpha$.

Example. The semilattice composition from Theorem 3 may not be strong. This is shown by the following example: let $Y = \{0,1,2\}$, $0 > 1 > 2$, be a semilattice, and let $S_\alpha = \{e_\alpha, a_\alpha\}$ be monoids in which the multiplication is given by $e_\alpha^2 = e_\alpha$, $e_\alpha a_\alpha = a_\alpha e_\alpha = a_\alpha^2 = a_\alpha$, $\alpha \in Y$. Define homomorphisms $\phi_{\alpha,\beta}$, $\alpha > \beta$, by

$$\phi_{0,1} = \begin{pmatrix} e_0 & a_0 \\ a_1 & a_1 \end{pmatrix}, \quad \phi_{0,2} = \begin{pmatrix} e_0 & a_0 \\ e_2 & a_2 \end{pmatrix}, \quad \phi_{1,2} = \begin{pmatrix} e_1 & a_1 \\ a_2 & a_2 \end{pmatrix},$$

$\phi_{\alpha,\alpha}$, $\alpha \in Y$, satisfying (1). Then $S = (Y\,; S_\alpha, \phi_{\alpha,\beta})$ *is a semilattice of monoids* S_α *and it is not a strong semilattice of monoids* S_α, since $(e_0\phi_{0,1})\phi_{1,2} \neq e_0\phi_{0,2}$.

Let $S = (B\,; S_i)$, where B is a band and each S_i is a monoid with the identity e_i. Then S is a *weakly systematic band of monoids* S_i *if for* $i, j, k \in B$, $i \geq j \geq k \Rightarrow e_ie_je_k = e_ie_k$.

By the following theorem we describe strong semilattices of matrices of monoids.

Theorem 4. *A semigroup* S *is a strong semilattice of matrices of monoids if and only if* S *is a weakly systematic normal band of monoids.*

Proof. Let $S = [Y\,; S_\alpha, \phi_{\alpha,\beta}]$. By Theorem 3 we obtain that S is a normal band of monoids. Let us use notations of Remark 1. Assume $i, j, k \in B$ such that $i \geq j \geq k$. Then $i \in B_\alpha$, $j \in B_\beta$, $k \in B_\gamma$, $\alpha \geq \beta \geq \gamma$ and $j = i\theta_{\alpha,\beta}$, $k = i\theta_{\alpha,\gamma}$, whence

$$e_ie_je_k = e_ie_{i\theta_{\alpha,\beta}}e_{i\theta_{\alpha,\gamma}} = e_ie_{i\theta_{\alpha,\beta}}e_{i\theta_{\alpha,\beta}\theta_{\beta,\gamma}} = e_i\phi_{\alpha,\beta}\phi_{\beta,\gamma} = e_i\phi_{\alpha,\gamma} = e_ie_{i\theta_{\alpha,\gamma}} = e_ie_k.$$

Conversely, let S be a weakly systematic normal band of monoids and let us use notations of Remark 1. If $\alpha, \beta, \gamma \in Y$, $\alpha \geq \beta \geq \gamma$ and $a \in S_i$, $i \in B_\alpha$, then

$$a\phi_{\alpha,\beta}\phi_{\beta,\gamma} = ae_{i\theta_{\alpha,\beta}}e_{i\theta_{\alpha,\beta}\theta_{\beta,\gamma}} = ae_ie_{i\theta_{\alpha,\beta}}e_{i\theta_{\alpha,\gamma}} = ae_ie_{i\theta_{\alpha,\gamma}} = a\phi_{\alpha,\gamma}.$$

Let $S = [Y; S_\alpha, \phi_{\alpha,\beta}]$, where $S_\alpha = (B_\alpha; S_i)$, B_α is a rectangular band and each S_i is a monoid with the identity e_i. S is a *special strong semilattice of* S_α if for $\alpha, \beta \in Y$, $\alpha \geq \beta \Rightarrow \{e_i \mid i \in B_\alpha\}\phi_{\alpha,\beta} \subseteq \{e_j \mid j \in B_\beta\}$ (M. Yamada [14]). In notations of Remark 1, this is equivalent with: $\alpha \geq \beta$, $i \in B_\alpha \Rightarrow e_ie_{i\theta_{\alpha,\beta}} = e_{i\theta_{\alpha,\beta}}$, $\alpha, \beta \in Y$, or, equivalently, if for $i, j \in B$, $i \geq j \Rightarrow e_ie_j = e_j$.

By Theorems 3 and 4 and Remark 1 we obtain the following consequences.

Corollary 1 [14]. *A semigroup S is a systematic normal band of monoids if and only if S is a special strong semilattice of systematic matrices of monoids.*

Corollary 2. *A semigroup S is a proper normal band of monoids if and only if S is a special strong semilattice of proper matrices of monoids.*

Note that S is a proper matrix of monoids if and only if S is isomorphic to a direct product of a monoid and a rectangular band [11].

Corollary 3. *A semigroup S is a normal band of unipotent monoids if and only if S is a strong semilattice of matrices of unipotent monoids.*

Corollary 4 [9]. *A semigroup S is a normal band of groups if and only if S is a strong semilattice of completely simple semigroups.*

Corollary 5 [9]. *A semigroup S is an orthodox normal band of groups if and only if S is a strong semilattice of rectangular groups.*

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References

[1] S. Bogdanović and M. Ćirić: Bands of monoids. Matem. Bilten, 9–10 (XXXV–XXXVI) (1985–1986); Skopje, 57–61 (1989).

[2] M. Ćirić and S. Bogdanović: Sturdy bands of semigroups. Collect. Math. Barcelona, **41** (3), 189–195 (1990).

[3] ——: Inflations of a band of monoids. Zb. rad. Fil. fak. Niš, Ser. Mat., **6**, 141–149 (1992).

[4] ——: Spined products of some semigroups (in preparation).

[5] A.H. Clifford: Semigroups admitting relative inverses. Annals of Math., (2) **42**, 1037–1049 (1941).

[6] G. Lallement: Demi-groupes réguliers. Ann. Mat. Pure Appl., (4) **77**, 47–129 (1967).

[7] M. Petrich: Regular semigroups satisfying certain conditions on idempotents and ideals. Trans. Amer. Math. Soc., **170**, 245–267 (1972).

[8] ——: The structure of completely regular semigroups. ibid., **189**, 211–236 (1974).

[9] ——: Introduction to Semigroups. Merill, Ohio (1973).

[10] ——: Lectures in Semigroups. Akad. Verlag, Berlin (1977).

[11] B.M. Schein: Bands of monoids. Acta Sci. Math. Szeged, **36**, 145–154 (1974).

[12] Š. Schwarz: Right compositions of semigroups. Math. Slovaca, **36** (1), 3–14 (1988).
[13] M. Yamada and N. Kimura: Note on idempotent semigroups. II. Proc. Japan Acad., **34**, 110–112 (1958).
[14] M. Yamada: Some remarks on strong semilattices of certain special semigroups. Math. Japonica, **33** (5), 813–820 (1988).
[15] R. Yoshida: *l*-compositions of semigroups. I. Mem. Res. Inst. Sci. Eng., Ritumeiken Univ., **14**, 1–12 (1965).
[16] ——: ditto. II. ibid., **15**, 1–5 (1966).