56. Flat Structure for the Simple Elliptic Singularity of Type $\tilde{E}_6$ and Jacobi Form

By Ikuo SATAKE

Department of Mathematics, Faculty of Science, Osaka University

(Communicated by Heisuke HIRONAKA, M. J. A., Sept. 13, 1993)

§1. Introduction. In order to construct the inverse mapping of the period mapping for the primitive form for the semi-universal deformation of a simple elliptic singularity, K. Saito introduced in [5] the "flat structure" for the extended affine root system. In section 3, we construct explicitly the flat theta invariants in the case of type E_6 using the Jacobi form introduced by Wirthmüller [7]. Combining the results of Kato [3], Noumi [4] (explicit description of the flat coordinates), this gives an answer to Jacobi's inversion problem (up to linear isomorphism) of this period mapping for a simple elliptic singularity of type $\tilde{E}_6$ (see also [6]). The details will be published elsewhere.

§2. Jacobi form. $\mathfrak{h}_{\boldsymbol{C}}$ is a (complexified) Cartan subalgebra for a fixed simple Lie algebra of rank l. $\mathfrak{h}_{\boldsymbol{C}}^* := Hom_{\boldsymbol{C}}(\mathfrak{h}_{\boldsymbol{C}}, \boldsymbol{C})$. $R^\vee$: the set of coroots. W: Weyl group. $Q(R^\vee)$: the $\boldsymbol{Z}$-span of $R^\vee$. $\langle , \rangle$: the Killing form normalized as $\langle \alpha, \alpha \rangle = 2$ for the highest root α. We identify $\mathfrak{h}_{\boldsymbol{C}}$ with $\mathfrak{h}_{\boldsymbol{C}}^*$ via $\langle , \rangle$. A symmetric tensor

$$\tilde{I}_W := \frac{\partial}{\partial \tau} \otimes \frac{\partial}{\partial t} + \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial \tau} + \sum_{i=1}^{l} \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_i}, \tag{2.1}$$

is defined on $\boldsymbol{H} \times \mathfrak{h}_{\boldsymbol{C}} \times \boldsymbol{C} \ni (\tau, z, t)$, where $\boldsymbol{H} := \{\tau \in \boldsymbol{C} ; Im\tau > 0\}$, z_i is an orthonormal basis of $\mathfrak{h}_{\boldsymbol{C}}^*$. The symbol $e(x)$ denotes $exp(2\pi\sqrt{-1}x)$.

Definition 2.1. *A Jacobi form of weight k and index m $(k, m \in \boldsymbol{Z})$ is a holomorphic function $\varphi : \boldsymbol{H} \times \mathfrak{h}_{\boldsymbol{C}} \times \boldsymbol{C} \to \boldsymbol{C}$ satisfying*

1) $\varphi(\tau, z + \lambda + \mu\tau, t - \frac{1}{2}\langle \mu, \mu \rangle \tau - \langle \mu, z \rangle) = \varphi(\tau, z, t)$ *for any* $\lambda, \mu \in Q(R^\vee)$,
2) $\varphi(\tau, w(z), t) = \varphi(\tau, z, t)$ *for any* $w \in W$,
3) $\varphi(\tau, z, t + \alpha) = e(-m\alpha)\varphi(\tau, z, t)$ *for any* $\alpha \in \boldsymbol{C}$,
4) $\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, t + \frac{c\langle z, z \rangle}{2(c\tau + d)}\right) = (c\tau + d)^k \varphi(\tau, z, t)$ *for any* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Z})$,
5) φ *has a Fourier series expansion of the form*
$$e(-mt) \sum_{n \in \boldsymbol{Z}} \phi_n(z) q^n \quad (q = e(\tau))$$
with $\phi_n(z) = 0$ *if* $n < 0$.

The vector space of all Jacobi forms of weight k and index m is denoted by $J_{k,m}$. Put

$$J_{**} = \bigoplus_{k,m \in \boldsymbol{Z}} J_{k,m}, \quad M_* = \bigoplus_{k \in \boldsymbol{Z}} J_{k,0}. \tag{2.2}$$

Theorem 2.2 (Wirthmüller [7]). *J_{**} is a polynomial algebra over M_*, freely generated by $l+1$ Jacobi forms $\varphi_0, \ldots, \varphi_l$ where φ_i is of weight k_i and index $m_i (m_0 \leq m_1 \leq \cdots \leq m_l)$ (except for the case of type E_8).*

Proposition 2.3. *For $\varphi \in J_{k,m}$ and $\phi \in J_{k',m'}$,*

$$\tilde{I}_W(d(\eta^{-2k}\varphi), d(\eta^{-2k'}\phi)) / \eta^{-2k-2k'} \in J_{k+k'+2,\, m+m'}, \tag{2.3}$$

where $\eta(\tau) := q^{1/24}\Pi_{n=1}^{\infty}(1-q^n)$ with $q = e(\tau)$.

We take $\varphi = \phi = \varphi_l$. Then

$$\psi := \tilde{I}_W(d(\eta^{-2k_l}\varphi_l), d(\eta^{-2k_l}\varphi_l)) / \eta^{-2k_l-2k_l} \tag{2.4}$$

is a Jacobi form of weight $2k_l + 2$ and index $2m_l$. Since $J_{2,0} = \{0\}$ by the basic theory of modular forms, any function multiplied by φ_l^2 does not appear when ψ is represented as a polynomial of $\varphi_0, \ldots, \varphi_l$ over M_*. Also the Jacobi forms $\varphi_0, \ldots, \varphi_l$ generate "the ring S^W" [5, p.34, (4.3.3)] over $\Gamma(\boldsymbol{H}, \mathcal{O}_H)$ [7]. This means that for the "codimension one case" $(i.e.\ m_i < m_l,\ i = 0, \cdots, l-1)$ [5, p.23, (2.4.2)], we can give "the normalized lowest degree vector field ∂_l" defined in [5, p.48, (9.4.1)] by

$$\frac{\partial}{\partial(\eta^{-2k_l}\varphi_l)}. \tag{2.5}$$

By this fact and Prop. 2.3, we can express "the tensor $J^* = \partial_l \tilde{I}_W$" [5, p.49, (9.6.1)] in terms of the Jacobi forms and the Dedekind η function.

§3. Jacobi forms and flat theta invariants of type E_6.

Theorem 3.1. (Wirthmüller [7]). *For the E_6 case, the ring J_{**} is a polynomial algebra over M_* on seven generators*

$\varphi_0 \in J_{0,1}$, $\varphi_1 \in J_{-2,1}$, $\varphi_2 \in J_{-5,1}$, $\varphi_3 \in J_{-6,2}$, $\varphi_4 \in J_{-8,2}$, $\varphi_5 \in J_{-9,2}$, $\varphi_6 \in J_{-12,3}$.

The existence of the above generators is shown by Wirthmüller [7].

Proposition 3.2. *These generators are unique under the following conditions.*

$$\lim_{q\to 0} \varphi_0 = 1/2e(-t)[S(\omega_1) + S(\omega_6) + 90], \tag{3.1}$$

$$\lim_{q\to 0} \varphi_1 = 1/2e(-t)[S(\omega_1) + S(\omega_6) - 54], \tag{3.2}$$

$$\lim_{q\to 0} \varphi_2 = e(-t)[S(\omega_1) - S(\omega_6)], \tag{3.3}$$

$$\begin{aligned}\lim_{q\to 0} \varphi_3 = e(-2t)[&-3/2[S(\omega_3) + S(\omega_5)] - 27S(\omega_2) + 5/8[S(\omega_1)^2 \\ &+ S(\omega_6)^2] + 3/4S(\omega_1)S(\omega_6) + 30[S(\omega_1) + S(\omega_6)] - 486],\end{aligned} \tag{3.4}$$

$$\begin{aligned}\lim_{q\to 0} \varphi_4 = e(-2t)[&-6[S(\omega_3) + S(\omega_5)] + 36S(\omega_2) + 3/2[S(\omega_1)^2 + \\ &S(\omega_6)^2] + S(\omega_1)S(\omega_6) - 60[S(\omega_1) + S(\omega_6)] + 324],\end{aligned} \tag{3.5}$$

$$\begin{aligned}\lim_{q\to 0} \varphi_5 = e(-2t)[&6[S(\omega_3) - S(\omega_5)] - [S(\omega_1)^2 - S(\omega_6)^2] \\ &- 42[S(\omega_1) - S(\omega_6)]],\end{aligned} \tag{3.6}$$

$$\begin{aligned}\lim_{q\to 0} \varphi_6 = e(-3t)[&108S(\omega_4) - 324S(\omega_2)] - 54S(\omega_2)[S(\omega_1) + S(\omega_6)] \\ &- 27/2[S(\omega_1)S(\omega_3) + S(\omega_5)S(\omega_6)] - 9/2[S(\omega_1)S(\omega_5) \\ &+ S(\omega_3)S(\omega_6)] \\ &- 162[S(\omega_3) + S(\omega_5)] + 15[S(\omega_1)^3 + S(\omega_6)^3] \\ &- 11S(\omega_1)S(\omega_6)[S(\omega_1) + S(\omega_6)]\end{aligned} \tag{3.7}$$

$$- 1620[S(\omega_1) + S(\omega_6)] - 666[S(\omega_1)^2 + S(\omega_6)^2] + 1836S(\omega_1)S(\omega_6) - 1944],$$

where ω_k is a fundamental weight (see Bourbaki [1]),

(3.8) $$S(\omega_k) := \sum_{w \in W/\text{ isotropy subgroup of } \omega_k} e(w \cdot \omega_k),$$

and $e(\omega_k)$ is a function $e(\omega_k) : \mathfrak{h}_{\boldsymbol{C}} \to \boldsymbol{C}$ defined by

(3.9) $$z \mapsto e(\langle z, \omega_k \rangle).$$

Theorem 3.3. *Let Θ_i be the functions satisfying the following relations:*

(3.10) $$\tilde{\varphi}_0 = F_1'\eta^2\Theta_0 + F_2'\eta^2\Theta_1,$$

(3.11) $$\tilde{\varphi}_1 = F_1\eta^2\Theta_0 + F_2\eta^2\Theta_1,$$

(3.12) $$\tilde{\varphi}_2 = \eta^2\Theta_2,$$

(3.13) $$\tilde{\varphi}_3 = F_3'\eta^2\Theta_3 + F_4'\eta^2\Theta_4 - \frac{3}{2\cdot 12^3}(\tilde{G}_3\tilde{\varphi}_0^2 - 2\tilde{G}_2^2\tilde{\varphi}_0\tilde{\varphi}_1 + \tilde{G}_2\tilde{G}_3\tilde{\varphi}_1^2),$$

(3.14) $$\tilde{\varphi}_4 = F_3\eta^2\Theta_3 + F_4\eta^2\Theta_4 + \frac{2}{12^3}(\tilde{G}_2\tilde{\varphi}_0^2 - 2\tilde{G}_3\tilde{\varphi}_0\tilde{\varphi}_1 + \tilde{G}_2^2\tilde{\varphi}_1^2),$$

(3.15) $$\tilde{\varphi}_5 = -2/3\eta^2\Theta_5,$$

(3.16) $$\begin{aligned}\tilde{\varphi}_6 = {} & \frac{-6}{2\pi\sqrt{-1}}\vartheta_6 \\ & + \frac{1}{12^3}[-6\tilde{G}_3\tilde{\varphi}_0\tilde{\varphi}_3 + 3/2\tilde{G}_2^2\tilde{\varphi}_0\tilde{\varphi}_4 + 6\tilde{G}_2^2\tilde{\varphi}_1\tilde{\varphi}_3 - 3/2\tilde{G}_2\tilde{G}_3\tilde{\varphi}_1\tilde{\varphi}_4] \\ & + \frac{1}{12^6}[-(5\tilde{G}_2^3 + 7\tilde{G}_3^2)\tilde{\varphi}_0^3 + 36\tilde{G}_2^2\tilde{G}_3\tilde{\varphi}_0^2\tilde{\varphi}_1 - 3\tilde{G}_2(5\tilde{G}_2^3 + 7\tilde{G}_3^2)\tilde{\varphi}_0\tilde{\varphi}_1^2 \\ & \qquad + \tilde{G}_3(8\tilde{G}_2^3 + 4\tilde{G}_3^2)\tilde{\varphi}_1^3],\end{aligned}$$

where

(3.17) $$\tilde{\varphi} := \eta^{-2k}\varphi \text{ for } \varphi \in J_{k,m},$$

(3.18) $$G_2 := (4\pi^4/3)^{-1}g_2(\tau) = (4\pi^4/3)^{-1}60 \sum_{m,n\in\boldsymbol{Z},(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^4} \in J_{4,0},$$

(3.19) $$G_3 := (8\pi^6/27)^{-1}g_3(\tau) = (8\pi^6/27)^{-1}140 \sum_{m,n\in\boldsymbol{Z},(m,n)\neq(0,0)} \frac{1}{(m\tau+n)^6} \in J_{6,0},$$

(3.20) $$\vartheta_6 := \Theta_6 - 2\frac{\eta'}{\eta}[\Theta_0\Theta_3 + \Theta_1\Theta_4 + \Theta_2\Theta_5],$$

(3.21) $$F_i' := \frac{-2}{\pi\sqrt{-1}}\eta^{-4}\frac{dF_i}{d\tau}\ (i = 1,2),\ F_i' := \frac{-3}{2\pi\sqrt{-1}}\eta^{-4}\frac{dF_i}{d\tau}\ (i = 3,4),$$

(3.22) $$F_i := f_i(z(\tau)) \left(z(\tau) := \frac{1}{2}\left[\frac{\sqrt{27}}{\sqrt{-1}}\frac{g_3(\tau)}{(2\pi)^6\eta(\tau)^{12}} + 1\right]\right),$$

$f_1(z)$ and $f_2(z)$ are solutions of Gauss' hypergeometric equation:

(3.23) $$z(1-z)\frac{d^2f(z)}{dz^2} + \left(\frac{2}{3} - \frac{4}{3}z\right)\frac{df(z)}{dz} + \frac{5}{12}f(z) = 0,$$

$f_3(z)$ and $f_4(z)$ are solutions of Gauss' hypergeometric equation:

(3.24) $$z(1-z)\frac{d^2f(z)}{dz^2} + \left(\frac{2}{3} - \frac{4}{3}z\right)\frac{df(z)}{dz} + \frac{1}{12}f(z) = 0,$$

satisfying

$$(3.25)\qquad \begin{bmatrix} F_1' & F_2' \\ F_1 & F_2 \end{bmatrix}\begin{bmatrix} F_3' & F_3 \\ F_4' & F_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{12}\tilde{G}_2^2 & \frac{1}{3}\tilde{G}_3 \\ \frac{1}{12}\tilde{G}_3 & \frac{1}{3}\tilde{G}_2 \end{bmatrix}.$$

Then the functions Θ_i are flat theta invariants [5] *and satisfy the following identity*:

$$(3.26)\qquad \frac{2}{(-2\pi\sqrt{-1})^2}\frac{\partial}{\partial\tilde{\varphi}_6}\tilde{I}_W(d\Theta_i, d\Theta_j)$$

$$= \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}$$

$$(i, j = -1, 0, \ldots, 6,\ \Theta_{-1} := \tau).$$

In the proof of Theorem 3.3, we use the following fact:

$$(3.27)\qquad \eta^{-4}\frac{2}{(-2\pi\sqrt{-1})^2}\frac{\partial}{\partial\tilde{\varphi}_6}\tilde{I}_W(d\tilde{\varphi}_i, d\tilde{\varphi}_j) =$$

$$\begin{pmatrix} 0&0&0&0&0&0&0&\frac{-6\eta^{-4}}{2\pi\sqrt{-1}}\\ 0&0&0&0&\frac{1}{12}\tilde{G}_2^2&\frac{1}{3}\tilde{G}_3&0&3\tilde{G}_2\tilde{\varphi}_1\\ 0&0&0&0&\frac{1}{12}\tilde{G}_3&\frac{1}{3}\tilde{G}_2&0&2\tilde{\varphi}_0\\ 0&0&0&0&0&0&-\frac{2}{3}&0\\ 0&\frac{1}{12}\tilde{G}_2^2&\frac{1}{12}\tilde{G}_3&0&\frac{1}{2}\tilde{G}_2\tilde{\varphi}_1&\frac{4}{3}\tilde{\varphi}_0&0&4\tilde{\varphi}_1^2+\frac{1}{2}\tilde{G}_2\tilde{\varphi}_4\\ 0&\frac{1}{3}\tilde{G}_3&\frac{1}{3}\tilde{G}_2&0&\frac{4}{3}\tilde{\varphi}_0&\frac{8}{3}\tilde{\varphi}_1&0&4\tilde{\varphi}_3\\ 0&0&0&-\frac{2}{3}&0&0&0&0\\ \frac{-6\eta^{-4}}{2\pi\sqrt{-1}}&3\tilde{G}_2\tilde{\varphi}_1&2\tilde{\varphi}_0&0&4\tilde{\varphi}_1^2+\frac{1}{2}\tilde{G}_2\tilde{\varphi}_4&4\tilde{\varphi}_3&0&* \end{pmatrix}$$

$$(i, j = -1, 0, \ldots, 6,\ \tilde{\varphi}_{-1} := \tau).$$

Remark. From the above results, we also obtain that

$$C\Theta_0 \oplus C\Theta_1 \oplus C\Theta_2 = C\eta^{-2}\chi_{\Lambda_0} \oplus C\eta^{-2}\chi_{\Lambda_1} \oplus C\eta^{-2}\chi_{\Lambda_6},$$

where Λ_0, Λ_1 and Λ_6 are fundamental weights of level 1 of $E_6^{(1)}$ type affine Lie algebra, χ_Λ is a normalized character of $E_6^{(1)}$ type (see [2, p.226]).

References

[1] Bourbaki, N.: Groupes et algèbres de Lie. Chapitres 4,5 et 6, Paris, Hermann (1969).

[2] Kac, V. G.: Infinite Dimensional Lie Algebras. 3rd ed., New York, Cambridge Univ. Press (1990).

[3] Kato, M.: The flat coordinates of universal unfoldings of $\tilde{E}_6$, $\tilde{E}_7$. Bull. Coll. Sci., Univ. Ryukyu, **42**, 5–10 (1986).

[4] Noumi, M.: Flat coordinate system of type E, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$. Tables (1986) (unpublished).

[5] Saito, K.: Extended affine root system. II. Publ. RIMS, **26**, 15–78 (1990).

[6] ——: Period mapping associated to a primitive form. ibid., **19** 1231–1264 (1983).

[7] Wirthmüller, K.: Root systems and Jacobi forms (1989) (preprint).