77. *Spectral Concentration and Resonances for Unitary Operators*

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1. Introduction. Operator-theoretical approach to the theory of resonances for a family of selfadjoint operators H_k has been investigated by J. S. Howland ([1]), A. Orth ([4]) and W. Hunziker ([2]). (For other works see the references in these papers, and [3; VIII, §5].) In particular, Orth established a link between the theory of resonances and the limiting absorption principle, developed the theory without any analyticity assumptions, and applied it successfully to N-body Schrödinger operators using the Mourre estimate.

In the present note we are mainly interested in the abstract part of the work [4] and shall present a generalization which can cover H_k given by a form sum. (Note that in [4] it is supposed that $H_k \supset H_0 + \kappa W$.) To this end we find it convenient to construct a counterpart of Orth's abstract results for a unitary operator family U_k. It will be given in §2. In §3 we transform the results to the selfadjoint families. This amounts to considering the Cayley transform $(H_\kappa - i)(H_\kappa + i)^{-1}$ of H_κ, or $(H_\kappa - d)^{-1}$ if H_κ is uniformly semibounded. In §4 we apply the results to a simple example in which a Dirichlet decoupled ordinary differential operator is perturbed by a delta type measure.

In this note we present only results. Detailed proofs will be published elsewhere ([5]).

The main instrument in [1] and [4] is the Livsic matrix. It is generally defined as follows.

Definition (L). Let T be a densely defined closed operator in a Hilbert space $\mathbf{H}$ and P be a finite dimensional orthogonal projection. Then the Livsic matrix $B(T, z)$ of T in $P\mathbf{H}$ is a finite dimensional operator defined by

$$P(T - z)^{-1}P = (B(z, T) - z)^{-1},$$

where z belongs to the resolvent set $\rho(T)$ of T.

2. Spectral concentration for unitary operators. Let U be a unitary operator and P be an orthogonal projection onto the m-dimensional space $K = P\mathbf{H}(m < \infty)$. It is not necessary that U and P commute. We put $\Omega_0 := \{w \in C ; |w| > 1\}$. We shall consider the Livsic matrix $B(w)$ of U in K. For $w \in \Omega_0$ $B(w)$ is well-defined and given as

$$B(w) = PUP - PU\overline{P}(\overline{U} - w)^{-1}\overline{P}UP$$

where $\overline{U} = \overline{P}U\overline{P}$.

Let $U_\kappa = \int_0^{2\pi} e^{i\theta}\, dF_\kappa(\theta)$, $\kappa \geq 0$, be unitary operators such that $U_\kappa \to U_0$ in the strong sense as $\kappa \to 0$. And let $e^{i\theta_0}$ be an eigenvalue of U_0

with finite multiplicity m. In the case of $m \geq 2$ we shall assume

$$U_\kappa = U_0 + a(\kappa) V_\kappa,$$

where $a(\kappa)$ is a complex continuous function such that, as $\kappa \to 0$, $a(\kappa)$ tends to zero and $\lim \arg a(\kappa)$ exists and V_κ is a bounded operator such that V_κ tends to V_0 in the strong sense. We denote by $B(z, \kappa)$ the Livsic matrix of U_κ in the eigenspace of U_0 corresponding to the eigenvalue $e^{i\theta_0}$. Corresponding to Definition 1.4 of [4] we shall introduce the following assumption.

Assumption (AU). *There exist a neighborhood* $C \subset [0, 2\pi]$ *of* θ_0 *and a complex neighborhood* Ω_1 *of* $e^{i\theta_0}$ *such that* $B(z, \kappa)$ *has a continuous extension from* $\Omega_0 \cap \Omega_1$ *to* e^{iC} *and the continuation satisfies*

$$\| B(z, \kappa) - B(w, \kappa) \| \leq L(\kappa) \mid z - w \mid$$

for $z, w \in \Omega := \overline{\Omega_0 \cap \Omega_1}$, *where we assume*

$$L(\kappa) = \begin{cases} o(1), & \text{if } m = 1, \\ o(a(\kappa)), & \text{if } m \geq 2 \end{cases}$$

Corresponding to Theorems 1.5 and 1.12 of [4] we have the following theorems.

Theorem 2.1. *Let* $e^{i\theta_0}$ *be a simple eigenvalue of* U_0 *and* ϕ *be an eigenvector corresponding to the eigenvalue* $e^{i\theta_0}$ *with* $\| \phi \| = 1$. *Suppose that the Livsic matrix* $B(z, \kappa)$ *of* U_κ *in the eigenspace* $\{\alpha\phi\}$ *satisfies Assumption (AU). Then, for sufficiently small* κ $(0 \leq \kappa \leq \kappa_0)$ *the following assertions* (1)-(3) *hold.*

(1) *There exists a unique solution of the equation*

$$z(\kappa) = (B(z(\kappa)/\mid z(\kappa) \mid, \kappa)\phi, \phi)$$

such that $\mid z(\kappa) \mid \leq 1$.

Put $z(\kappa) = r(\kappa) e^{i\theta(\kappa)}$.

(2) *There exists* $\delta(\kappa) \geq 0$ *such that* $\delta(\kappa) = 0$ *if* $r(\kappa) = 1$ *and if* $r(\kappa) < 1$ *and* $\kappa \to 0$, *then*

$$\max(\delta(\kappa), L(\kappa)^{1/2} \delta(\kappa)/(1 - r(\kappa)), (1 - r(\kappa))/\delta(\kappa)) \to 0 \text{ as } \kappa \to 0.$$

(3) *For any* $\delta(\kappa)$ *in* (2) *put* $C(\kappa) = [\theta(\kappa) - \delta(\kappa), \theta(\kappa) + \delta(\kappa)]$. *Then*

$$F_\kappa(C(\kappa)) \to P, \ \kappa \to 0,$$

in the strong sense.

Theorem 2.2. *Let* $e^{i\theta_0}$ *be a degenerate eigenvalue with a finite multiplicity* m *of the unitary operator* U_0. *Suppose that the Livsic matrix of the unitary operators satisfy Assumption* (AU) *and that* PV_0P *has only simple eigenvalues* $\mu_1, \ldots, \mu_m$. *Then the following assertions* (1)-(3) *hold.*

(1) *We can find the unique solutions* $z_1(\kappa), \ldots, z_m(\kappa)$ *of*

$$\det(B(z/\mid z \mid, \kappa) - z) = 0$$

satisfying $\mid z_j(\kappa) - e^{i\theta_0} - a(\kappa)\mu_j \mid = o(a(\kappa))$. *We put*

$$z_j(\kappa) = r_j(\kappa) e^{i\theta_j(\kappa)} \text{ and } B_j(\kappa) = B(e^{i\theta_j(\kappa)}, \kappa).$$

(2) *There exists* $\delta_j(\kappa) \geq 0$ *such that, for* $r_j(\kappa) = 1$, $\delta_j(\kappa) = 0$, *and for* $r_j(\kappa) < 1$,

$$\max(\delta_j(\kappa)/\mid a(\kappa) \mid, L(\kappa)^{1/2} \delta_j(\kappa)/\mid a(\kappa) \mid^{1/2} (1 - r_j(\kappa)), (1 - r_j(\kappa))/\delta_j(\kappa)) \to 0, \text{ as } \kappa \to 0.$$

(3) *Let* $C_j(\kappa) = [\theta_j(\kappa) - \delta_j(\kappa), \theta_j(\kappa) + \delta_j(\kappa)]$ *and* $P_j(0)$ *be the projection associated to the eigenvalue* $z_j(0)$ *of* $B_j(0)$. *Then we have*

$$F_\kappa(C_j(\kappa)) \rightarrow P_j(0),\ \kappa \rightarrow 0,$$

in the strong sense.

3. Application to the selfadjoint problems. We shall consider the following situation. H_κ, $\kappa \geq 0$, is a selfadjoint operator in $\mathbf{H}$. H_κ converges to H_0 in the strong resolvent sense. λ_0 is an eigenvalue of H_0 with a finite multiplicity. Let P be the orthogonal projection associated to the eigenvalue λ_0 of H_0, $K=$ Range P, $m=\dim K<\infty$ and $\overline{P}:=I-P$.

In this section we use a linear fractional function $g(H_\kappa)$ of H_κ. We shall first assume

Assumption (G.1). $g(z)=(az+b)/(cz+d)$, $a, b, c, d \in C$ with $ad-bc \neq 0$. *There exists* $\kappa_0>0$ *such that* $c\lambda+d \neq 0$ *for any* $\lambda \in \cup_{0<\kappa<\kappa_0}\sigma(H_\kappa)$.

We shall write the Livsic matrix $B(g(z), g(H_\kappa))$ in K as $B_g(g(z),\kappa)$, that is

$$P(g(H_\kappa)-g(z))^{-1}P=(B_g(g(z),\kappa)-g(z))^{-1},\ Imz \neq 0.$$

If $K \subset D(g(H_\kappa))$, we can write $B_g(g(z),\kappa)$ as

$$B_g(g(z),\kappa)=Pg(H_\kappa)P-Pg(H_\kappa)\overline{P}(\overline{g}(H_\kappa)-g(z))^{-1}\overline{P}g(H_\kappa)P,$$

where $\overline{g}(H_\kappa)=\overline{P}g(H_\kappa)\overline{P}$. In the case that $m \geq 2$ we assume in addition to Assumption (G.1) the following condition:

Assumption (G.2). $g(H_\kappa)=g(H_0)+a(\kappa)V_\kappa$
where $a(\kappa) \neq 0$, $\kappa \neq 0$, *is a complex valued continuous function with* $a(\kappa) \rightarrow 0$ *as* $\kappa \rightarrow 0$ *with* $\lim \arg a(\kappa)$ *existing and* V_κ *are bounded operators such that* $V_\kappa \rightarrow V_0$ *as* $\kappa \rightarrow 0$. *And* PV_0P *has only simple eigenvalues* $\mu_1,\ldots,\mu_m$.

Assumption (AG). *There exist a real neighborhood* I *of* λ_0 *and a complex neighborhood* Ω *of* λ_0 *such that* $B_g(g(z),\kappa)$ *has a continuous extension from* $C\setminus R$ *to* I *and the continuation satisfies*

$$\| B_g(g(z),\kappa)-B_g(g(w),\kappa)\| \leq L(\kappa)\,|z-w|$$

for any $z, w \in \Omega$, *where*

$$L(\kappa)=\begin{cases} o(1), & \text{if } m=1, \\ o(a(\kappa)), & \text{if } m \geq 2 \end{cases}$$

We shall define the resonances as follows.

Definition 1 (simple resonance). We call λ_0 *a simple resonance* of the operator family $\{H_\kappa\}$, if λ_0 is the simple eigenvalue of H_0 and if there exists a function g satisfying (G.1) and the Livsic matrix of $g(H_\kappa)$ satisfies Assumption (AG).

Definition 2 (resonance). We call λ_0 a *resonance* of the operator family $\{H_\kappa\}$, if there exists a function g such that g satisfies (G.1), $g(H_\kappa)$ satisfies (G.2) and the Livsic matrix of $g(H_\kappa)$ satisfies (AG).

Then we have the following theorems. E_κ is the spectral resolution of H_κ.

Theorem 3.1. *Let* λ_0 *be a simple resonance of the operator family* $\{H_\kappa\}$. *Then there exist closed intervals* $J(\kappa)$ *approaching* λ_0 *such that the length of* $J(\kappa)$ *tends to* 0 *and* $E_\kappa(J(\kappa))$ *converges to* P *in the strong sense as* $\kappa \rightarrow 0$.

Theorem 3.2. *Let* λ_0 *be a resonance of the operator family of* $\{H_\kappa\}$. *Then* *there exist closed intervals* $J_j(\kappa)$ *approaching* λ_0 *such that the length of* $J_j(\kappa)$ *tend to* 0 *and* $E_\kappa(J_j(\kappa)) \to P_j$ as $\kappa \to 0$ *where* P_j *is the projection onto the* μ_j*-associated eigenspace of* PV_0P.

In the next theorem we only consider the case of $g(z) = (z-i)/(z+i)$ for simplicity.

Theorem 3.3. *Let* λ_0 *be a resonance of the operator family of* H_κ *and* $\phi \in K$. *Then we have for* $g(z) = (z-i)/(z+i)$, $0 \leq \kappa \leq \kappa_0$ *and* $t \geq 0$:
(1) *Simple case;*

$$(\exp(-itH_\kappa)\phi, \phi) = \exp(-iz(\kappa)t)\,\|\phi\|^2 + o(1),$$

where $z(\kappa)$ *is the solution of*

$$g(z) = B_g(g(z)/|g(z)|, \kappa).$$

(2) *Degenerate case;*

$$(\exp(-itH_\kappa)\phi, \phi) = \textstyle\sum_{j=1}^m \|P_j\phi\|^2 \exp(-iz_j(\kappa)t) + o(1),$$

where $z_j(\kappa)$ *is the solution of*

$$\det(B_g(g(z)/|g(z)|, \kappa) - g(z)) = 0$$

and satisfies $|g(z_j(\kappa)) - g(\lambda_0) - a(\kappa)\mu_j| = o(a(\kappa))$.

These theorems generalize Theorems 1.5, 1.12, 1.8 and 1.14 of [4].

4. An Application. We shall consider the following second order ordinary differential operators on a half-line $[0, \infty)$, one with the Dirichlet condition at $x = 1$ and the other with a "jump condition" there. In this example H_κ is defined as a form sum.

$$\text{(I)} \qquad \begin{cases} H_0 u = -\dfrac{d^2}{dx^2} u \text{ on } L^2(0, \infty), \\ u(0) = u(1 \pm 0) = 0. \end{cases}$$

$$\text{(II)} \qquad \begin{cases} H_\kappa u = -\dfrac{d^2}{dx^2} u \text{ on } L^2(0, \infty), \\ u(0) = 0,\ u(1-0) = u(1+0) \equiv u(1), \\ u'(1+0) - u'(1-0) = \dfrac{1}{\kappa} u(1),\ \kappa > 0. \end{cases}$$

It is well-known that H_0 has embedded eigenvalues $\{m^2\pi^2\}_{m \geq 1}$ and a continuous spectrum $[0, \infty)$. Then we expect that these embedded eigenvalues are resonances.

Theorem 4.1. *Let* $g(z) = 1/(z+1)$ *and* φ_m *be a normalized eigenfunction of* H_0 *corresponding to the eigenvalue* $\lambda_m = m^2\pi^2$, *i.e.,* $\varphi_m(x) = \sqrt{2}\sin m\pi x$ *for* $0 \leq x \leq 1$, $= 0$ *for* $1 < x$. *Let* P *be the orthogonal projection onto the eigenspace* $\{\alpha\varphi_m\}_{\alpha \in C}$. *Then*

(1) g *satisfies Assumption* (G.1) *and* $g(H_\kappa) = (H_\kappa + 1)^{-1}$ *satisfies Assumption (AG). In particular, the spectral concentration as in Theorem* 3.1 *occurs.*

(2) *Furthermore we have*

$$|(\exp(-itH_\kappa)\varphi_m, \varphi_m)| = \exp(-4m^3\pi^3\kappa^2 t) + o(1).$$

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