72. Nonconvex-valued Differential Inclusions in a Separable Hilbert Space

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1. Introduction. Let $\mathfrak{B}$ be a real separable Hilbert space which is densely and continuously imbedded in another real separable Hilbert space $\mathfrak{H}$. A correspondence (= multi-valued mapping) $\Gamma : [0, T] \times \mathfrak{B} \twoheadrightarrow \mathfrak{H}$ is assumed to be given. We consider the differential inclusion:

$$\dot{x} \in \Gamma(t, x),\ x(0) = 0.$$

Maruyama [7] examined a differential inclusion of this type in the case $\mathfrak{B} = \mathfrak{H}$ and established the existence of solutions under rather restrictive assumptions. In particular the following two assumptions are not satisfied in many important situations:

(i) the correspondence $x \mapsto\!\!\!\!\rightarrow \Gamma(t, x)$ is upper hemi-continuous with respect to the weak topology for $\mathfrak{B}$ and the strong topology for $\mathfrak{H}$, and

(ii) the correspondence Γ is convex-valued.

The first assumption can be weakened to:

(i′) the correspondence $x \mapsto\!\!\!\!\rightarrow \Gamma(t, x)$ is upper hemi-continuous with respect to the weak topologies for both of $\mathfrak{B}$ and $\mathfrak{H}$,

under the additional assumption that Γ is bounded. (See section 6.)

However it seems quite hard to drop the second assumption without any serious change of the proof. In fact, Maruyama [8] exemplified the importance of assumption (ii) in deducing several properties of differential inclusions including the existence of solutions.

In this paper, we shall show the way leading to the existence without having recourse to assumption (ii). Examples will be shown in section 5.

2. Assumptions. We begin by specifying some assumptions imposed on the correspondence $\Gamma : [0, T] \times \mathfrak{B} \twoheadrightarrow \mathfrak{H}$. Denote by $\mathfrak{B}^w$ (resp. $\mathfrak{H}^w$) the space $\mathfrak{B}$ (resp. $\mathfrak{H}$) endowed with the weak topology.

Assumption 1. *The set $\Gamma(t, x) \subset \mathfrak{H}^w$ is nonempty and weakly compact for all $(t, x) \in [0, T] \times \mathfrak{B}$.*

Assumption 2. *For each fixed $t \in [0, T]$, the correspondence $x \mapsto\!\!\!\!\rightarrow \Gamma(t, x)$ is continuous with respect to the weak topologies for both of $\mathfrak{B}$ and $\mathfrak{H}$; i.e. Γ satisfies both the upper hemi-continuity and the lower hemi-continuity in x.* (For the concept of "continuity" of a correspondence, see Aubin-Frankowska [1] Chap. 1.)

Assumption 3. *For each fixed $x \in \mathfrak{B}$, the correspondence $t \mapsto\!\!\!\!\rightarrow \Gamma(t, x)$ is measurable in the sense that the weak inverse image $\Gamma^{-w}(U) = \{t \in [0, T] : \Gamma(t, x) \cap U \neq \emptyset\}$ is measurable for all open sets U in $\mathfrak{H}^w$ and for each fixed $x \in \mathfrak{B}$.* (For the concept of "measurability" of a correspondence, see Aubin-Frankowska [1] Chap. 8.)

Assumption 4. *There exists* $M > 0$ *such that*

$$\sup\{\|y\| : y \in \Gamma(t, x),\ x \in \mathfrak{B},\ t \in [0, T]\} \leqq M.$$

3. Notations and lemmas. Let $B_{\mathfrak{B}}$ (resp. $B_{\mathfrak{H}}$) be the closed ball in $\mathfrak{B}$ (resp. $\mathfrak{H}$) with center at zero and radius MT (resp. M). Since both $\mathfrak{B}$ and $\mathfrak{H}$ are real separable Hilbert spaces, the topology of the closed ball $B_{\mathfrak{B}}$ (resp. $B_{\mathfrak{H}}$) induced by the weak topology of $\mathfrak{B}$ (resp. $\mathfrak{H}$) is completely metrizable. Let us denote such a metric by $d_{\mathfrak{B}}$ (resp. $d_{\mathfrak{H}}$). Without loss of generality, we may assume that $d_{\mathfrak{B}}$ satisfies the condition:

(1) $$d_{\mathfrak{B}}(x, x') \leqq \|x - x'\|_{\mathfrak{B}} \quad \text{for every } x, x' \in B_{\mathfrak{B}},$$

where $\|\cdot\|_{\mathfrak{B}}$ is the usual norm in $\mathfrak{B}$. We denote by h the Hausdorff distance defined, for any subsets B, B' of $B_{\mathfrak{H}}$ by

$$h(B, B') = \mathrm{Max}\{\sup_{x\in B}\inf_{y\in B'} d_{\mathfrak{H}}(x, y),\ \sup_{x\in B'}\inf_{y\in B} d_{\mathfrak{H}}(x, y)\}.$$

Let us introduce the function $\eta : [0, T] \times R_+ \to R_+$, called the modulus of continuity of Γ, defined by

$$\eta(t, r) = \mathrm{Max}\{h(\Gamma(t, x), \Gamma(t, x')) : x, x' \in B_{\mathfrak{B}},\ d_{\mathfrak{B}}(x, x') \leqq r\}.$$

Filippov's measurable implicit function theorem implies that η is measurable in t for each fixed r, and from Berge's maximum theorem we deduce that η is continuous in r for each fixed t.

Let $\mathfrak{R} \equiv \mathfrak{R}([0, T], B_{\mathfrak{H}})$ be the space of all measurable functions of $[0, T]$ into $B_{\mathfrak{H}}$. Let d be the metric on $\mathfrak{R}$ defined by

$$d(u, v) = \int_0^T d_{\mathfrak{H}}(u(t), v(t))\, dt\ ;\ u, v \in \mathfrak{R}.$$

Lemma 1. *The metric space* $(\mathfrak{R}, d)$ *is complete.*

Proof. Let $\{u_n\}$ be a Cauchy sequence in $\mathfrak{R}$. Then there exists a subsequence $\{u_{n(k)}\}$, $n(1) < n(2) < \cdots < n(k) < \cdots$, such that

$$\mu\{t \in [0, T] : d_{\mathfrak{H}}(u_{n(k+1)}(t), u_{n(k)}(t)) > 1/2^k\} < 1/2^k,$$

where μ is the usual Lebesgue measure on $[0, T]$. If we set $E_k = \{t : d_{\mathfrak{H}}(u_{n(k+1)}(t), u_{n(k)}(t)) > 1/2^k\}$ and $E = \limsup_k E_k$, then it is easy to check that $\mu(E) = 0$. Furthermore if $t \notin E$, then there exists $k_0 > 0$ such that

$$d_{\mathfrak{H}}(u_{n(k+1)}(t), u_{n(k)}(t)) \leqq 1/2^k \text{ for all } k \geqq k_0.$$

Therefore, for $t \notin E$, $\{u_{n(k)}(t)\}$ constitutes a Cauchy sequence in $B_{\mathfrak{H}}$.

We denote the limit of $u_{n(k)}(t)$ by $u(t) \in B_{\mathfrak{H}}$ and set

$$u^*(t) = \begin{cases} u(t) & \text{if } t \notin E \\ 0 & \text{if } t \in E. \end{cases}$$

Then we obtain $d(u_n, u^*) \to 0$ as $n \to \infty$ since

$$d(u_n, u^*) \leqq d(u_n, u_{n(k)}) + d(u_{n(k)}, u^*),$$

the right hand side of which tends to zero as n and $n(k)$ tend to ∞. Q.E.D.

Let r_i be a decreasing sequence of real numbers tending to 0. Since $B_{\mathfrak{B}}$ is a weakly compact set which is metrizable, it is totally bounded. Hence there exists, for each r_i, a finite subset $\mathscr{A}_i = \{a_1, a_2, \cdots, a_{n(i)}\}$ of $B_{\mathfrak{B}}$ satisfying:

(2) $$B_{\mathfrak{B}} = \bigcup_{j=1}^{n(i)} \{x \in B_{\mathfrak{B}} : d_{\mathfrak{B}}(x, a_j) < r_i/4\}.$$

We define a sequence $\mathscr{B}_i \subset \mathscr{A}_1 \times \mathscr{A}_2 \times \cdots \times \mathscr{A}_i (i = 1,2,\cdots)$ of sets inductively. For $i = 1$, put $\mathscr{B}_1 = \mathscr{A}_1$. Now suppose that $\mathscr{B}_{i-1}$ has been defined and put

$$\mathscr{B}_i = \{b : b = (a^1, a^2, \cdots, a^{i-1}, a^i), (a^1, \cdots, a^{i-1}) \in \mathscr{B}_{i-1}, \\ a^i \in \mathscr{A}_i, \text{ and } d_{\mathfrak{B}}(a^{i-1}, a^i) \leqq r_{i-1}\}.$$

Now we assign, for each $b = (a^1, \cdots, a^i)$, an integrable function $u_b : [0, T] \to \mathfrak{H}$ which satisfies the following (3) and (4):

(3) $u_b(t) = u_{(a^1,\ldots,a^i)}(t) \in \Gamma(t, a^i)$ a.e. in $[0, T]$,

(4) $d_{\mathfrak{H}}(u_b(t), u_{(a^1,\ldots,a^{i-1})}(t)) \leqq \eta(t, r_{i-1})$ a.e. in $[0, T]$ $(i > 1)$.

The existence of such a function u_b can be shown by an inductive reasoning as follows. The intersection

$$\Delta(t) \equiv \Gamma(t, a^i) \cap \{x : d_{\mathfrak{H}}(x, u_{(a^1,\ldots,a^{i-1})}(t)) \leqq \eta(t, r_{i-1})\}$$

is not empty for each $t \in [0, T]$ since $d_{\mathfrak{B}}(a^{i-1}, a^i) \leqq r_{i-1}$. Furthermore the correspondence $t \mapsto\!\!\!\to \Delta(t)$ is closed-valued and measurable, thus it has a measurable selection $u_b(t)$, which is integrable by Assumption 4.

Let h_i be a decreasing sequence of real numbers tending to 0 such that T/h_1, and each h_i/h_{i+1} are integers. We define $\mathscr{C}_i$ to be the set of functions $c : [0, T] \to \mathscr{B}_i$ such that each of the component function $a^j(t) (1 \leqq j \leqq i)$ of $c(t) = (a^1(t), \cdots, a^i(t))$ is constant on intervals $[s h_j, (s+1)h_j)$ for each integer $s = 0,1,\cdots,T/h_j - 1$. It is clear from Assumption 4 and (3) that

$$u_{c(t)}(t) \in B_{\mathfrak{H}} \text{ for } t \in [0, T]. \quad \text{for each } c \in \mathscr{C}_i,\ i = 1,2,\cdots.$$

Consider now the set $\mathscr{U}_i = \{u : [0, T] \to \mathfrak{H} ; u = u_c \text{ for some } c \in \mathscr{C}_i\}$. Since $\mathscr{C}_i$ is finite and u_b is uniquely defined for each $b \in \mathscr{B}_i$, $\mathscr{U}_i$ is finite.

We denote $\mathscr{U} = \bigcup_{i \in N} \mathscr{U}_i$.

Lemma 2. *If*

$$\int_0^T \sum_{i=1}^{\infty} \eta(t, r_i)\, dt < +\infty, \tag{5}$$

then $\mathscr{U}$ is sequentially compact in $(\mathfrak{K}, d)$.

Proof. Since $\mathfrak{K}$ is complete by lemma 1, it is sufficient to prove that $\mathscr{U}$ is totally bounded. Take an arbitrary $\varepsilon > 0$. By (5), we can take $m \in N$ such that

$$\int_0^T \sum_{i=m}^{\infty} \eta(t, r_i)\, dt < \varepsilon. \tag{6}$$

We shall prove that for each $u \in \mathscr{U}_{m+p}$, $p \geqq 1$, there exists $u' \in \mathscr{U}_m$ such that $d(u, u') \leqq \varepsilon$.

If $u(t) = u_{(a^1(t),\ldots,a^m(t),\ldots,a^{m+p}(t))}(t)$, then we put $u'(t) = u_{(a^1(t),\ldots,a^m(t))}(t)$. It is clear that $u' \in \mathscr{U}_m$. We have:

$$d_{\mathfrak{H}}(u_{(a^1(t),\ldots,a^{m+p}(t))}(t), u_{(a^1(t),\ldots,a^m(t))}(t)) \\ \leqq \sum_{i=1}^{p} d_{\mathfrak{H}}(u_{(a^1(t),\ldots,a^{m+i}(t))}(t), u_{(a^1(t),\ldots,a^{m+i-1}(t))}(t)).$$

Using (4) and (6), we obtain

$$d(u, u') \leqq \int_0^T \sum_{i=1}^{p} \eta(t, r_{m+i-1}) \leqq \varepsilon.$$

Q.E.D.

4. Main theorem. Theorem 1. *Under Assumptions* 1, 2, 3, *and* 4, *there exists an absolutely continuous function* $x : [0, T] \to \mathfrak{B}$ *with* $x(0) = 0$ *such that*

$$\dot{x}(t) \in \Gamma(t, x(t)) \quad \textit{for a.e. } t \textit{ in } [0, T].$$

Proof. We construct a sequence $\{x_n(t)\}$ of approximate solutions which satisfies the following four conditions:

(i) $x_n : [0, T] \to \mathfrak{B}$ is absolutely continuous.

(ii) $x_n(0) = 0$.

(iii) There exists $c \in \mathscr{C}_n$ such that

$$\dot{x}(t) = u_{c(t)}(t) \quad \text{for each } t \in [0, T].$$

(iv) If $c(t) = (a^1(t), a^2(t), \cdots, a^n(t))$, then

$$d_{\mathfrak{B}}(x_n(t), a^n(t)) \leqq r_n \quad \text{for } t \in [0, T].$$

For this purpose, let us assume that $\{r_n\}$ and $\{h_n\}$ defined in Section 3 also satisfy (5) and the following:

(7) $$r_{n+1} < r_n/4, \ n = 1,2,\cdots,$$

(8) $$h_n M < r_n/4, \ n = 1,2,\cdots.$$

To construct such a function x_n, we need to specify $c(t) = (a^1(t), \cdots, a^n(t))$ so that it satisfies the above mentioned conditions (i)-(iv). We use the induction argument on the partition of $[0, T]$ of length h_n.

For $t \in [0, h_n)$ we put

$$a^i(t) = a^i = \text{constant},$$

where $a^i \in \mathscr{A}_i$ and $d_{\mathfrak{B}}(a^i, 0) \leqq r_i/4$, $i = 1,2,\cdots,n$. Such a^i exists by (2).

Suppose that $c(t)$ has been defined for $t \in [0, s h_n]$, $1 \leqq s < T/h_n$. For each fixed $i \leqq n$, we define $a^i(t)$ for $t \in [s h_n, (s+1)h_n)$ by:

Case 1. $s h_n \neq p h_i$ for each integer p. Then

$$a^i(t) = a^i((s-1)h_n) \quad \text{for } t \in [s h_n, (s+1)h_n).$$

Case 2. There is an integer p such that $s k_n = p k_i$. Then

$$a^i(t) = a^i = \text{constant} \quad \text{for } t \in [s h_n, (s+1)h_n),$$

where $a^i \in \mathscr{A}_i$ and $d_{\mathfrak{B}}(a^i, x_n(s h_n)) \leqq r_i/4$. Such a^i exists by (2).

It is easy to check that for each i and each integer $p < T/h_i$, $a^i(t)$ is constant on $[p h_i, (p+1) h_i)$ and

(9) $$d_{\mathfrak{B}}(x_n(s h_n), a^i(s h_n)) < r_i/2, \ i = 1,2,\cdots,n.$$

The inequality (9) in the first case follows from (1) and (8). From (7) and (9), we see that the function c takes its values in $\mathscr{B}_n$. Therefore $c \in \mathscr{C}_n$. (iv) follows easily from (1), (8), and (9).

By definition, each x_n is an absolutely continuous function such that $\dot{x}_n \in \mathscr{U}$. Thus by Lemma 2, $\{\dot{x}_n\}$ has a subsequence, without changing notations for the sake of simplicity, which converges in $(\mathfrak{K}, d)$ to a measurable function $v \in \mathfrak{K}$. Thus $\{\dot{x}_n\}$ has a subsequence, again by using the same notation, such that

(10) $\dot{x}_n(t) \to v(t)$ weakly a.e. $t \in [0, T]$ in $\mathfrak{H}^w$.

Furthermore, when we consider each x_n as an element of the Sobolev space $\mathfrak{W}^{1,1}([0, T], \mathfrak{B})$, there exists a subsequence, again without changing notations, and some $x^* \in \mathfrak{W}^{1,1}([0, T], \mathfrak{B})$ such that

(11) $x_n \to x^*$ uniformly in $\mathfrak{B}^w$ on $[0, T]$, and

(12) $\dot{x}_n \to \dot{x}^*$ weakly in $\mathfrak{L}^1([0, T], \mathfrak{B})$,

thanks to Maruyama's convergence theorem (Maruyama [7], Theorem 1). By (12) and Mazur's theorem, there exists a sequence of convex combinations $\{\sum_{i=1}^{k_n} \alpha_i^n \dot{x}_{n+i}\}$ which converges strongly to $\dot{x}^*$ in $\mathfrak{L}^1$. Hence, by taking a subsequence, say $\{z_n\}$, we get:

$$z_n(t) \to \dot{x}^*(t) \text{ a.e. } t \in [0, T] \text{ in } \mathfrak{B}.$$

If we consider each $z_n(t)$ and $\dot{x}^*(t)$ as a point in $\mathfrak{H}$, we get:

$$z_n(t) \to \dot{x}^*(t) \text{ a.e. } t \in [0, T] \text{ in } \mathfrak{H}.$$

Thus we conclude, by virtue of (10), that

(13) $$\dot{x}^*(t) = v(t) \text{ a.e. on } [0, T].$$

To prove that x^* is a solution, we note that:

$$\begin{aligned} d_{\mathfrak{H}}(\dot{x}_n(t), \Gamma(t, x_n(t))) &= d_{\mathfrak{H}}(u_{(a^1(t),\dots,a^n(t))}(t), \Gamma(t, x_n(t))) \\ &\leqq d_{\mathfrak{H}}(u_{(a^1(t),\dots,a^n(t))}(t), \Gamma(t, a_n(t))) \\ &\qquad + h(\Gamma(t, a_n(t)), \Gamma(t, x_n(t))). \end{aligned}$$

From (3), the first component of the above sum is equal to zero. Thus, from (iv) in this proof, we have the inequality

$$d_{\mathfrak{H}}(\dot{x}_n(t), \Gamma(t, x_n(t))) \leqq \eta(t, r_n),$$

which, together with (10), (11), (13) and the continuity of Γ in x, implies

$$d_{\mathfrak{H}}(\dot{x}^*(t), \Gamma(t, x^*(t))) = 0 \quad \text{a.e. } t \text{ in } [0, T].$$

Hence

$$\dot{x}^*(t) \in \Gamma(t, x^*(t)) \quad \text{a.e. } t \text{ in } [0, T]. \qquad \text{Q.E.D.}$$

5. Examples. In this section, we shall provide some examples. Let $\mathfrak{A}$ be a topological space. One of the problems we have in mind is the following control problem;

$$\dot{x}(t) = f(t, x(t), u(t)),$$
$$u(t) \in U(t),$$

where $f : [0, T] \times \mathfrak{B}^w \times \mathfrak{A} \to \mathfrak{H}^w$ is assumed to be measurable in t and continuous in x and u, and $U : [0, T] \twoheadrightarrow \mathfrak{A}$ is a compact-valued measurable correspondence. If we define $\Gamma : [0, T] \times \mathfrak{B} \twoheadrightarrow \mathfrak{H}$ by

$$\Gamma(t, x) = \{f(t, x, u) : u \in U(t)\},$$

then Γ is easily seen to satisfy Assumptions 1, 2, and 3. Finally we shall give a function which satisfies the weak-weak continuity. Let $g : R^{mn} \to R^n$ be a continuous function. We consider the function $G : \mathfrak{W}^{m,2}([0, T], R^n) \to \mathfrak{L}^2([0, T], R^n)$ defined by:

$$G(x)(t) = g(x(t), Dx(t), \cdots, D^{m-1}x(t)) \,;\, x \in \mathfrak{W}^{m,2}.$$

G is continuous with respect to the weak topologies for both of $\mathfrak{W}^{m,2}$ and $\mathfrak{L}^2$ by Sobolev's lemma.

6. Final remarks. We considered, in this paper, a differential inclusion of Filippov's type defined in a real separable Hilbert space. We are greatly indebted to Kaczyński-Olech [6] and Maruyama [7] for the methods embodied in the proof. The key reasoning of the proof is essentially the same as Kaczyński-Olech, and we owe Maruyama for the treatment of infinite dimensional spaces.

Maruyama informed me in a private communication the following result, the proof of which is essentially the same as Maruyama [7], Theorem 2.

Theorem. *Let $\mathfrak{H}^w$ be a real separable Hilbert space endowed with the weak topology. Assume that $\Gamma : [0, T] \times \mathfrak{H}^w \twoheadrightarrow \mathfrak{H}^w$ satisfies the following conditions:*

(i) *Γ is compact-convex-valued.*

(ii) *The correspondence $x \mapsto\!\!\!\rightarrow \Gamma(t, x)$ is upper hemi-continuous.*

(iii) *The correspondence $t \mapsto\!\!\!\rightarrow \Gamma(t, x)$ is measurable.*

(iv) *There exists $M > 0$ such that*

$$\sup \{\|y\| : y \in \Gamma(t, x),\ t \in [0, T],\ x \in \mathfrak{H}\} \leqq M.$$

Then there exists $x^* \in \mathfrak{W}^{1,2}([0, T], \mathfrak{H})$ such that

$$\dot{x}^*(t) \in \Gamma(t, x^*) \text{ a.e. in } [0,T].$$

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