69. *Improvement in the Irrationality Measures of π and π^2*

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§0. In this note we show the new results concerning lower bounds for rational approximations to π, π^2 and some other numbers involving π. These bounds will be derived from particular integrals of some rational functions involving Legendre type polynomials.

W. M. Schmidt [18] stated that a Roth type theorem should hold for the classical constants in analysis such as π, π^2, log 2, $\zeta(3)$, $\cdots$, as Lang had said that it should hold for any "reasonably" defined number. Note that it does hold for almost all transcendental numbers. Our results in this note are, however, still far from the conjecture but these effective results can be considered as a step to this direction.

§1. The following lemma due to F. Beukers [2] is very important in the study of rational approximations to $\zeta(2)=\pi^2/6$. Let D_n be the least common multiple of $\{1,2,\cdots,n\}$. For any polynomial $P(x)$ let $\deg(P)$ and $\mathrm{ord}(P)$ be the degree and the order of zero point at the origin of $P(x)$ respectively. Put $S=[0,1]\times[0,1]$.

Lemma 1.1. *For any polynomials $f(z)$ and $g(z)$ with integral coefficients, we have*

$$\int_S\int \frac{f(x)g(y)}{1-xy}\,dx\,dy = a\,\zeta(2)+b,$$

where

$$a=\frac{1}{2\pi i}\int_C f(z)g\left(\frac{1}{z}\right)\frac{dz}{z}$$

is an integer (C denotes a closed curve enclosing the origin) and b is a rational number whose denominator is a divisor of D_ND_M with $M=\max\{\deg(f),\deg(g)\}$ and

$$N=\min\{\max\{\deg(f),\deg(g)-\mathrm{ord}(f)\},\ \max\{\deg(g),\deg(f)-\mathrm{ord}(g)\}\}.$$

We now consider the following double integral:

$$\varepsilon_n=\int_S\int\frac{(x(1-x))^{15n}(y(1-y))^{14n}}{(1-xy)^{12n+1}}\,dx\,dy \tag{1}$$

for any integer $n\geq 1$. After k-fold and $(12n-k)$-fold partial integrations with respect to x and y respectively, it follows that

$$\binom{12n}{k}\varepsilon_n=\int_S\int\frac{F_k(x)G_k(y)}{1-xy}\,dx\,dy \tag{2}$$

for any $k\in[0,12n]$, where

$$F_k(x) = \frac{1}{k!} x^{k-12n} (x^{15n}(1-x)^{15n})^{(k)},$$

$$G_k(y) = \frac{1}{(12n-k)!} (y^{14n-k}(1-y)^{14n})^{(12n-k)}.$$

Then, applying Lemma 1.1 to the integral (2), we obtain a rational approximation to $\zeta(2)$. Moreover the denominator of the b in Lemma 1.1 in this case is fairly small, since the integral coefficients of Legendre type polynomials $F_k(x)$ and $G_k(y)$ have a large common factor. Thus we have

Theorem 1.2. *There exists a positive integer q_0 such that*

$$\left| \pi^2 - \frac{p}{q} \right| \gg q^{-6.3489}$$

for all $p \in \mathbf{Z}$ and any integer $q \geq q_0$.

In other words, π^2 has an *irrationality measure* less than 6.3489. This improves the earlier results: 10.02979, 7.552, 7.5252 and 7.325 obtained by R. Dvornicich and C. Viola [9], E. A. Rukhadze [17], the author [10], and by D. V. Chudnovsky and G. V. Chudnovsky[5,6] respectively.

The above theorem also implies the following

Corollary 1.3. *For each integer $k \geq 1$, $\pi/\sqrt{k}$ has an irrationality measure less than* 12.6978.

This gives a good irrationality measure of π, since it improves the earlier measures: 30, 20, 19.8899944 and 13.394 obtained by K. Mahler [14], M. Mignotte [15], G. V. Chudnovsky [3] and by the author [11] respectively. These results were derived from the classical approximation formulae to exponential functions due to Hermite. However we can find better irrationality measures of $\pi/\sqrt{k}$ for some particular integral values of k by a different way.

§2. To investigate further rational approximations to π, we next introduce the following complex integral:

$$\int_\Gamma \frac{((z-a_1)(z-a_2)(z-a_3))^{2n}}{z^{3n+1}} dz \tag{3}$$

instead of the real integral (1), where a_1, a_2, a_3 are non-zero distinct complex numbers and Γ is a smooth path departing from a_j and arriving at a_k through a saddle of the surface defined by the function $|((z-a_1)(z-a_2)(z-a_3))^2/z^3|$ for some $j, k \in [1, 3]$. The asymptotic behaviour of the integral (3), as n tends to $+\infty$, can be easily obtained by *the saddle method* originated in Riemann's work. (See, for example, J. Dieudonné [7].)

By taking $(a_1, a_2, a_3) = (1, 2, 1+i)$, the integral (3) enables us to obtain the following

Theorem 2.1. *There exists a positive integer H_0 such that*

$$|p + q\pi + r\log 2| \gg H^{-7.0161}$$

for any integers p, q, r satisfying $H \equiv \max\{|q|, |r|\} \geq H_0$.

In other words, 1, π and log 2 have a *linear independence measure* less than 7.0161. In particular, π has an irrationality measure less than 8.0161, which remarkably improves the earlier results stated in Section 1. Of course, the number $\pi/\log 2$ has also an irrationality measure less than 8.0161.

§3. Corollary 1.3 in the case $k=3$ does not give a sharp result. Indeed, the better measures: 8.30998, 5.7926, 5.516, 5.0874 and 4.97 were already obtained by K. Alladi and M. L. Robinson [1], G. V. Chudnovsky [4], A. K. Dubitskas [8], the author [10] and by G. Rhin [16] respectively. However our integral (3) in the case $(a_1, a_2, a_3) = \left(1, \frac{1+\sqrt{3}i}{2}, \frac{3+\sqrt{3}i}{4}\right)$ enables us to improve the above results slightly as follows:

Theorem 3.1. *There exists a positive integer* q_1 *such that*

$$\left|\frac{\pi}{\sqrt{3}} - \frac{p}{q}\right| \gg q^{-4.6016}$$

for all $p \in \boldsymbol{Z}$ *and any integer* $q \geq q_1$.

The integral (3) can be also used to show that $\pi/\sqrt{3}\log 3$ and $\pi/\sqrt{3}\log\left(\frac{4}{3}\right)$ have irrationality measures less than 9.3853 and 8.8138 respectively. The theorems stated in this note are proved in the manuscripts [12, 13], which will be published in other journals.

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