68. On the Pro-p Gottlieb Theorem

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The purpose of this note is to present a remark on center-triviality of certain pro-p groups. We shall show the following

Theorem 1. *Let p be a rational prime, G a pro-p group, and $\boldsymbol{F}_p$ the trivial G-module of order p. Suppose that the following three conditions are satisfied.*

(1) $$cd_p G = n < \infty,$$

(2) $$H^i(G, \boldsymbol{F}_p) \text{ is finite for } i \geq 0,$$

(3) $$\sum_i (-1)^i \dim H^i(G, \boldsymbol{F}_p) \neq 0.$$

Then each open subgroup of G has trivial centralizer in G. In particular, the center of G is trivial.

Observing that the conditions (1)–(3) are inherited by any open subgroup of G, we see that we may prove just the center-triviality of G. The proof is divided into two steps.

Step 1. Let $\Lambda = \boldsymbol{Z}_p[[G]]$ be the complete group algebra of G over the ring of p-adic integers $\boldsymbol{Z}_p$. Then Λ is a local pseudocompact ring whose unique open maximal ideal R is the kernel of the canonical augmentation $\Lambda \to \boldsymbol{Z}/p\boldsymbol{Z}$. The following 'Nakayama lemma' due to A. Brumer [1] plays a crucial role in this step.

Lemma 2 (Brumer). *Let Λ be a pseudocompact ring with radical R, M a pseudocompact Λ-module, and let $x_1, \ldots, x_m \in M$. If M/RM is* (*topologically*) *generated by the images of $x_1, \ldots, x_m$, then $M = \Lambda x_1 + \cdots + \Lambda x_m$.*

Proof. See [1] Corollary 1.5.

It is remarkable that, in contrast to the usual Nakayama lemma, the above Brumer's lemma does not assume the finite generation of M as a Λ-module, but does imply it.

Lemma 3. *Let G be a pro-p group satisfying the conditions* (1),(2) *of Theorem* 1. *Then the trivial Λ-module $\boldsymbol{Z}_p$ has a finite free resolution :*

$$(F): \quad 0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to \boldsymbol{Z}_p \to 0,$$

where each F_i is a free Λ-module of finite rank $(0 \leq i \leq n)$.

Proof. We shall follow an argument in Gruenberg [3] 8.1 carefully in our context.

1°. We first show by induction on $N \geq 1$ that there is an exact sequence of Λ-modules

$$(A_N): \quad 0 \to K_N \to F_{N-1} \to \cdots \to F_0 \to \boldsymbol{Z}_p \to 0,$$

in which F_i $(0 \leq i \leq n-1)$ are free of finite ranks and K_N is arbitrary. If $N = 1$, then we can take as $F_0 = \Lambda$, $K_1 =$ the augmentation ideal of Λ. So we assume that the exact sequence (A_N) is obtained. To obtain (A_{N+1}), it suffices to show that K_N in the sequence (A_N) is finitely generated. As the category of pseudocompact Λ-modules is enough projective [1], we get the exact sequence

$$Hom_G(F_{N-1}, F_p) \to Hom_G(K_N, F_p) \to Ext_\Lambda^N(Z_p, F_p) \to 0.$$

Here Hom_G denotes the continuous G-homomorphisms. Moreover, by [1] Lemma 4.2, we have $Ext_\Lambda^N(Z_p, F_p) = H^N(G, F_p)$. Since the first and the third terms are finite, $Hom_G(K_N, F_p)$ turns out to be a finite set. From this the finiteness of K_N/RK_N follows. Thus, by Brumer's lemma, K_N is finitely generated over Λ.

2°. From 1°, we get an exact sequence

$$\cdots \to F_{n+1} \to F_n \to F_{n-1} \to \cdots \to F_0 \to Z_p \to 0$$

with F_i free Λ-module of finite rank ($i \geq 0$). We claim that $K := Image(F_n \to F_{n-1})$ is a projective object as a pseudocompact Λ-module. For this, it suffices to show that for every short exact sequence $0 \to A \to B \to C \to 0$ of finite Λ-modules, the induced map $Hom_G(K, B) \to Hom_G(K, C)$ is surjective ([1] Proposition 3.1). We have an exact sequence

$$\begin{array}{ccccccc} Hom_G(F_{n-1}, B) & \to & Hom_G(K, B) & \to & H^n(G, B) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ Hom_G(F_{n-1}, C) & \to & Hom_G(K, C) & \to & H^n(G, C) & \to & 0, \end{array}$$

and the first and the third vertical arrows are surjective. (Use the condition (1) of Theorem 1 for the latter.) Thus our assertion follows.

3°. Since the above K is obviously finitely generated, it remains to show the freeness of K. We can choose $x_1, \ldots, x_m \in K$ such that the images $\bar{x}_i (i = 1, \ldots m)$ in K/RK form a basis. Then by Brumer's lemma, we get

$$K = \Lambda x_1 + \cdots + \Lambda x_m.$$

Define F_n to be $\bigoplus_{i=1}^m \Lambda x_i$, and let Q be the kernel of the canonical projection $F \to K$. Then since K is projective, F is Λ-isomorphic to $K \oplus Q$. Therefore

$$F/RF \cong K/RK \oplus Q/RQ.$$

Comparing the dimensions ($< \infty$), we get $Q/RQ = 0$. It follows from Brumer's lemma again that $Q = 0$. Thus K is free of finite rank.

Step 2. We next apply the argument of J. Stallings [6] in our profinite context. This method was previously considered in [5] for giving a simple criterion for center-freeness of certain profinite fundamental groups of algebraic varieties (see Remark 2 below).

We first begin by an arbitrary profinite group G. For an open normal subgroup U of G and a positive integer a, let $T(Z/p^aZ[G/U])$ denote the quotient of the group algebra $Z/p^aZ[G/U]$ by the submodule generated by the $xy - yx$ ($x, y \in Z/p^aZ[G/U]$). Then the canonical projections

$$T_{U,a} : Z/p^aZ[G/U] \to T(Z/p^aZ[G/U]) \quad (n > 0,\ U \lhd G : \text{open})$$

form an inverse system of surjections of finite abelian groups. Taking the inverse limit, we obtain a profinite abelian group $T(Z_p[[G]])$ together with a continuous surjective homomorphism

$$T : Z_p[[G]] \to T(Z_p[[G]])$$

such that $T(\lambda + \mu) = T(\lambda) + T(\mu)$, $T(\lambda\mu) = T(\mu\lambda)$ for $\lambda, \mu \in Z_p[[G]]$. Each element of $T(Z_p[[G]])$ may be viewed as a Z_p-valued measure on the space of the conjugacy classes of G. (See [5] §1.3 for a little more leisured construction of the 'profinite Hattori-Stallings space' $T(\boldsymbol{Z}_p[[G]])$.) We let $\Lambda = \boldsymbol{Z}_p[[G]]$.

Definition. Let P be a finitely generated projective pseudocompact Λ-module and let $f: P \to P$ be a Λ-endomorphism. We define the *profinite Hattori-Stallings trace* $tr(f) \in T(\Lambda)$ as follows. Choose a pseudocompact Λ-module Q with $P \oplus Q \cong \Lambda^{\oplus m}$ for some $m \in \boldsymbol{N}$, and let $\bar{f} = (f, 0): \Lambda^{\oplus m} \to \Lambda^{\oplus m}$ be the 0-extension of f. Let $(\bar{f}_{ij}) \in M_n(\Lambda)$ be the matrix representation of $\bar{f}$ and define $tr(f) = \sum_{i=1}^{m} T(\bar{f}_{ii})$. It is easy to see the well-definedness of $tr(f)$ and the properties $tr(f+g) = tr(f) + tr(g)$, $tr(fg) = tr(gf)$ for two Λ-endomorphisms f, g of P.

Lemma 4. *Let G be a profinite group, and p be a prime number. Suppose that the trivial Λ-module $\boldsymbol{Z}_p$ has a finite free resolution*

$$(F): 0 \to F_n \to \cdots \to F_1 \to F_0 \to \boldsymbol{Z}_p \to 0,$$

where $F_i (1 \leq i \leq n)$ are finitely generated free Λ-modules, with Euler characteristic $\chi := \sum(-1)^i rank(F_i) \neq 0$. Then G has trivial center.

Proof. This is Theorem 1.3.2 of [5]. We repeat the proof briefly for the convenience of the reader, which is just a profinite modification of Stallings [6]. Let γ be any central element of G, and consider two Λ-endomorphisms (f_i), (g_i) of the complex (F) such that $f_i =$ identity and $g_i =$ multiplication by γ on F_i for $i = 0, \ldots, n$. By standard argument in homology theory, we can construct a chain homotopy between (f_i) and (g_i) to obtain

$$0 = \sum(-1)^i tr(f_i) - \sum(-1)^i tr(g_i) = \chi \cdot (\delta_1 - \delta_\gamma).$$

Here δ_1 (resp. δ_γ) is the Dirac measure supported at the conjugacy class $\{1\}$ (resp. $\{\gamma\}$). Since $\chi \neq 0$, and since the (profinite) space of the conjugacy classes of G is Hausdorff, we get $\gamma = 1$.

Proof of Theorem 1. By Lemmas 3 and 4, we may just assure

$$\sum_i (-1)^i rank(F_i) = \sum_i (-1)^i dim H^i(G, \boldsymbol{F}_p).$$

But we know by [1] Lemma 4.2 that $H^i(G, \boldsymbol{F}_p) = Ext^i_\Lambda(\boldsymbol{Z}_p, \boldsymbol{F}_p)$. Here Ext^i_Λ is the i-th extension group in the category of pseudocompact Λ-modules, and can be computed from the resolution in Lemma 3 in the usual way. Thus our assertion certainly follows.

Remark 1. A. Pletch (J. Pure and Appl. Algebra, 16) showed the existence of a finite free $\boldsymbol{Z}_p[[G]]$-resolution of $\boldsymbol{Z}_p$ for certain profinite groups G including pro-p groups, in which some delicate arguments on duality theory (due to P. Gabriel) were involved. In our proof of Lemma 3 above, we restricted ourselves to the case of pro-p groups, and modified part of Pletch's argument along [3] to be able to avoid delicate discussion on duality theory.

Remark 2. The existence of finite free resolution of $\boldsymbol{Z}_p$ can be also assured for profinite groups isomorphic to the pro-$\mathfrak{C}$ completion of $\mathfrak{C}$-good groups of type FL ($\mathfrak{C}$: a 'full' class of finite groups containing $\boldsymbol{Z}/p\boldsymbol{Z}$). See [5] 1.3.3.

Application to Galois groups. Let k be a number field of finite degree over the rationals, S a set of places of k containing those lying over a prime p. Denote by $k_S(p)$ the maximal pro-p extension of k unramified outside S, and by G_S the Galois group $Gal(k_S(p)/k)$. Then it is known that the Euler characteristic of G_S is equal to $-r_2$ (r_2: the number of complex places of k). Therefore, by the above theorem, if k is not totally real, then $G_S(p)$ has trivial center. This gives an alternative proof of Theorem 2.2 (1) of [7]. In [7], Yamagishi also considered the totally real case, and showed that in that case the centerfreeness of G_S is naturally related with the Leopoldt conjecture.

References

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