66. *On the Uniform Distribution Modulo One of Some Log-like Sequences*

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1. Introduction and main results. Let p_n denote the nth prime number. Let f be a polynomial with real coefficients, then it is known that the sequence $\{f(p_n)\}_{n=1}^{\infty}$ is uniformly distributed modulo one (u.d. mod 1) if and only if f is an irrational polynomial, which means that the polynomial $f(x)-f(0)$ has one irrational coefficient at least. (cf. [3]). Furthermore, it is also known that for any noninteger $\alpha \in (0, \infty)$, the sequence $\{p_n^{\alpha}\}_{n=1}^{\infty}$ is u.d. mod 1 (see e.g. [1], [6]).

On the other hand, Goto and Kano [2] investigated the log-like functions f and obtained sufficient conditions on the function f for which the sequence $\{f(p_n)\}_{n=1}^{\infty}$ is u.d. mod 1. Unfortunately we could not underestand the proof of main Theorem 2. In this paper we first modify Goto and Kano's results (see Theorems 1 and 2 below) and then give a new result (Theorem 3). The proofs are given in Section 2. (Though our Theorem 1 is essentially the same as Theorem 1 of [2], we give here a proof for completeness' sake.)

Theorem 1. Let $a > 0$ and let $f : [a, \infty) \to (0, \infty)$ be a differentiable function. Assume that $xf'(x) \to \infty$ as $x \to \infty$ and that for sufficiently large x, $(\log x) f'(x)$ is monotone in x. Further, assume that for some $\varepsilon > 0$, $f(x) = o((\log x)^{\varepsilon})$ as $x \to \infty$. Then the sequence $\{\alpha f(p_n)\}_{n=n_0}^{\infty}$ is u.d. mod 1, where $n_0 = min\{n : p_n > a\}$ and α is any nonzero real constant.

Theorem 2. Let $a > 0$ and let $f : [a, \infty) \to (0, \infty)$ be a twice differentiable function with $f' > 0$. Assume that $x^2 f''(x) \to \infty$ as $x \to \infty$ and that for sufficiently large x, $(\log x)^2 f''(x)$ is nonincreasing in x. Further, assume that for some $\varepsilon > 0$, $f(x) = o((\log x)^{\varepsilon})$ as $x \to \infty$. Then the sequence $\{\alpha f(p_n)\}_{n=n_0}^{\infty}$ is u.d. mod 1, where $n_0 = min\{n : p_n > a\}$ and α is any nonzero real constant.

Theorem 3. Let $a > 0$ and let $f : [a, \infty) \to (0, \infty)$ be a twice differentiable function with $f' > 0$. Assume that $x^2 f''(x) \to -\infty$ as $x \to \infty$ and that for sufficiently large x, both $(\log x)^2 f''(x)$ and $x(\log x)^2 f''(x)$ are nondecreasing in x. Further, assume that for some $\varepsilon > 0$, $f(x) = o((\log x)^{\varepsilon})$ as $x \to \infty$. Then the sequence $\{\alpha f(p_n)\}_{n=n_0}^{\infty}$ is u.d. mod 1, where $n_0 = min\{n : p_n > a\}$ and α is any nonzero real constant.

Note that Theorem 2 is essentially concerned with a convex function f, while Theorem 3 is concerned with a concave function f. Applying Theorem 3 to the function $f(x) = (\log x)^{\varepsilon}$ we obtain that the sequence $\{(\log p_n)^{\varepsilon}\}_{n=1}^{\infty}$ is u.d. mod 1 if $\varepsilon > 1$.

2. The proofs. We first prove Theorem 3 and then prove Theorems 1 and 2.

Proof of Theorem 3. By Weyl criterion (see e.g. [3] p.4) it suffices to prove that the sequence $\{f(p_n)\}_{n=n_0}^{\infty}$ is u.d. mod 1.

Since $x^2 f''(x) \to -\infty$ as $x \to \infty$, $f''(x) < 0$ for sufficiently large x. Without loss of generality, we may assume that for all $x \geq a$, $f''(x) < 0$ and that both the functions $(\log x)^2 f''(x)$ and $x(\log x)^2 f''(x)$ are nondecreasing in $x \in [a, \infty)$. To prove the uniform distribution modulo one of the sequence $\{f(p_n)\}_{n=n_0}^{\infty}$, we shall prove that the discrepancy D_N of $\{f(p_n)\}_{n=n_0}^{\infty}$ approaches zero as $N \to \infty$ (see e.g. [3] pp. 88–89). Actually, we shall prove that under the monotonicity conditions on the functions $(\log x)^2 f''(x)$ and $x(\log x)^2 f''(x)$,

$$(1) \quad D_N = O(\{f(p_N)(\log p_N)^{-\varepsilon}\}^{\frac{1}{2}} + \{-p_N^2 f''(p_N)\}^{-\frac{1}{2}} + \{-p_N^2(\log p_N) f''(p_N)\}^{-1}) \text{ as } N \to \infty,$$

which approaches zero as $N \to \infty$ due to the conditions $x^2 f''(x) \to -\infty$ and $f(x) = o((\log x)^{\varepsilon})$ as $x \to \infty$.

It remains to prove (1). As usual, we shall apply the Erdös-Turán's estimation of the discrepancy D_N of $\{f(p_n)\}_{n=n_0}^{N}$: for any positive integer m, there exists an absolute constant C such that

$$(2) \quad D_N \leq C\left(\frac{1}{m} + \sum_{h=n_0}^{m} \frac{1}{h}\left|\frac{1}{N}\sum_{n=n_0}^{N} e^{2\pi i h f(p_n)}\right|\right)$$

(see e.g. [3] p.114). The crucial point is to estimate the exponential sum in (2). Denote $q_0 = (p_{n_0} + a)/2$ and denote the sum

$$(3) \quad S_{n_0,N,h} = \sum_{n=n_0}^{N} e^{2\pi i h f(p_n)}.$$

Then using integration by parts we can rewrite (3) as follows:

$$(4) \quad S_{n_0,N,h} = \pi(p_N) e^{2\pi i h f(p_N)} - \pi(q_0) e^{2\pi i h f(q_0)} - \int_{q_0}^{p_N} (L^*(x) + R^*(x))\, d\, e^{2\pi i h f(x)},$$

where $\pi(x)$ is the number of primes not exceeding x, $\int_a^b = \int_{(a,b]}$, $R^*(x) = \pi(x) - L^*(x)$ and $L^*(x) = \int_{q_0}^{x} (\log t)^{-1}\, dt$ for $x \geq q_0$. The last integral in (4) is equal to

$$L^*(p_N) e^{2\pi i h f(p_N)} - L^*(q_0) e^{2\pi i h f(q_0)} - \int_{q_0}^{p_N} (\log x)^{-1} e^{2\pi i h f(x)} dx + 2\pi i h \int_{q_0}^{p_N} R^*(x) f'(x) e^{2\pi i h f(x)} dx.$$

Hence the exponential sum defined in (3) can be rewritten as

$$(5) \quad S_{n_0,N,h} = \{R^*(p_N) e^{2\pi i h f(p_N)} - R^*(q_0) e^{2\pi i h f(q_0)}\} + \int_{q_0}^{p_N} (\log x)^{-1} e^{2\pi i h f(x)} dx - 2\pi i h \int_{q_0}^{p_N} R^*(x) f'(x) e^{2\pi i h f(x)} dx \equiv I_1 + I_2 + I_3 \text{ (say)}.$$

We now estimate each I_i, $i = 1, 2, 3$. It follows from the Prime Number Theorem of Hadamard and de la Vallée-Poussin (see e.g. [5] chapter 3) that

$$(6) \quad R^*(x) = O(x(\log x)^{-k}) \text{ for each } k > 1.$$

Applying (6) to the estimations of I_1 and I_3 yields

$$(7) \quad |I_1| = O(p_N(\log p_N)^{-(1+\varepsilon)}) \text{ as } N \to \infty$$

and, since $f' > 0$,

(8) $$|I_3| = O(p_N h f(p_N)(\log p_N)^{-(1+\varepsilon)}) \text{ as } N \to \infty.$$

On the other hand, using Lemma 10.3 [7] (p. 225) we obtain that

(9) $$|I_2| \le \max_{q_0 \le x \le p_N} (4\{(\log x)\,|hf''(x)|^{\frac{1}{2}}\}^{-1} + \{x(\log x)^2\,|hf''(x)|\}^{-1})$$
$$= O(\{(\log p_N)(-hf''(p_N))^{\frac{1}{2}}\}^{-1} + \{p_N(\log p_N)^2(-hf''(p_N))\}^{-1}) \text{ as } N \to \infty,$$

in which the last equality follows from the monotonicity condition on the functions $(\log x)^2 f''(x)$ and $x(\log x)^2 f''(x)$.

Note that $|I_1| = O(|I_3|)$ as $N \to \infty$ because $f' > 0$. Putting (5), (7), (8) and (9) into (2) yields that for any positive integer m,

(10) $$D_N \le C\Big(\frac{1}{m} + \sum_{h=n0}^{m} \frac{1}{Nh}(|I_1| + |I_2| + |I_3|)\Big)$$

(11) $$= O\Big(\frac{1}{m} + \frac{1}{N}\{(\log p_N)(-f''(p_N))^{\frac{1}{2}}\}^{-1} + \frac{1}{N}\{p_N(\log p_N)^2(-f''(p_N))\}^{-1} + \frac{m}{N} p_N f(p_N)(\log p_N)^{-(1+\varepsilon)}\Big) \text{ as } N \to \infty.$$

Taking $m = \{N(\log p_N)^{1+\varepsilon}/(p_N f(p_N))\}^{\frac{1}{2}}$ in (11) and using $N \sim p_N/\log p_N$ as $N \to \infty$, we conclude that

$$D_N = O(\{f(p_N)(\log p_N)^{-\varepsilon}\}^{\frac{1}{2}} + \{-p_N^2 f''(p_N)\}^{-\frac{1}{2}} + \{-p_N^2(\log p_N) f''(p_N)\}^{-1}) \text{ as } N \to \infty,$$

which is the desired result (1). The proof is complete.

Proof of Theorem 1. Since $xf'(x) \to \infty$ as $x \to \infty$, $f'(x) > 0$ for sufficiently large x. Without loss of generality, we may assume that for all $x \ge a$, $f'(x) > 0$ and that the function $(\log x)f'(x)$ is monotone in $x \in [a, \infty)$. As before, to prove that the discrepancy D_N of $\{f(p_n)\}_{n=n_0}^{\infty}$ approaches zero as $N \to \infty$, we estimate each I_i defined in (5). The estimations of I_1 and I_3 are the same as those in (7) and (8), respectively. As to the estimation of I_2, we apply Lemma 4.3 of [7] (p. 61) and obtain that

(12) $$|I_2| = O(h^{-1} max\{1, [(\log p_N) f'(p_N)]^{-1}\}) \text{ as } N \to \infty,$$

because the function $(\log x)^2 f'(x)$ is monotone in x. It follows from (7), (8), (10) and (12) that

(13) $$D_N = O\Big(\frac{1}{m} + max\Big\{\frac{1}{N}, [p_N f'(p_N)]^{-1}\Big\} + \frac{m}{N} p_N f(p_N)(\log p_N)^{-(1+\varepsilon)}\Big) \text{ as } N \to \infty.$$

Taking $m = \{N(\log p_N)^{1+\varepsilon}/(p_N f(p_N))\}^{\frac{1}{2}}$ in (13) we obtain that

$$D_N = O\Big(\{f(p_N)(\log p_N)^{-\varepsilon}\}^{\frac{1}{2}} + max\Big\{\frac{1}{N}, [p_N f'(p_N)]^{-1}\Big\}\Big) \text{ as } N \to \infty,$$

which approaches zero as $N \to \infty$ due to the conditions $xf'(x) \to \infty$ and $f(x) = o((\log x)^{\varepsilon})$ as $x \to \infty$. The proof is complete.

Proof of Theorem 2. Since $x^2 f''(x) \to \infty$ as $x \to \infty$, $f''(x) > 0$ for sufficiently large x. Without loss of generality, we may assume that for all $x \ge a$, $f''(x) > 0$ and that the function $(\log x)^2 f''(x)$ is nonincreasing in $x \in [a, \infty)$. As before, we want to prove that the discrepancy D_N of $\{f(p_n)\}_{n=n_0}^{N}$ approaches zero as $N \to \infty$. The estimations of I_1 and I_3 defined in (5) are the same as those in (7) and (8), respectively. As to the estimation of I_2, we apply Lemma 10.2 of [7] (p. 225) and obtain that

$$(14) \qquad |I_2| \leq 4 \max_{q_0 \leq x \leq p_N} \{(\log x)(hf''(x))^{\frac{1}{2}}\}^{-1}$$
$$= 4\{(\log p_N)(hf''(p_N))^{\frac{1}{2}}\}^{-1},$$

in which the last equality follows from the condition that function $(\log x)^2 f''(x)$ is nonincreasing in x. Therefore, it follows from (7), (8), (10) and (14) that

$$(15) \quad D_N = O\left(\frac{1}{m} + \{p_N^2 f''(p_N)\}^{-\frac{1}{2}} + \frac{m}{N} p_N f(p_N)(\log p_N)^{-(1+\varepsilon)}\right) \text{ as } N \to \infty.$$

Taking $m = \{N(\log p_N)^{1+\varepsilon}/(p_N f(p_N))\}^{\frac{1}{2}}$ in (15) we obtain that

$$D_N = O(\{f(p_N)(\log p_N)^{-\varepsilon}\}^{\frac{1}{2}} + \{p_N^2 f''(p_N)\}^{-\frac{1}{2}}) \text{ as } N \to \infty,$$

which approaches zero as $N \to \infty$ due to the condition $x^2 f''(x) \to \infty$ and $f(x) = o((\log x)^{\varepsilon})$ as $N \to \infty$. The proof is complete.

References

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