65. *An Application of a Theorem of Rudin*

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1. Introduction. The study of generalized paracompact spaces has become significant in recent years. In addition to the new results in this area there have been a number of new interesting questions that have arisen from these studies. In this paper we answer one of these questions by applying a significant theorem of M.E. Rudin [7].

Definition 1. A family $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ is *closure-preserving* if for every subset $B \subseteq A$,

$$\bigcup_{\beta\in B} \overline{F_\beta} = \overline{\bigcup_{\beta\in B} F_\beta}.$$

Likewise, $\mathcal{F}$ is *hereditarily closure-preserving* if for every $B \subseteq A$ and $\{H_\beta : \beta \in B\}$ where $H_\beta \subseteq F_\beta$, $\bigcup_{\beta\in B} \overline{H_\beta} = \overline{\bigcup_{\beta\in B} H_\beta}$.

Let P be one of the following properties; discrete (D), locally finite (LF), hereditarily closure-preserving (HCP), and closure-preserving (CP). The symbol λ will denote any countable ordinal.

Definition 2. A space X is *$B(P, \lambda)$-refinable* provided every open cover $\mathcal{U}$ of X has a refinement $\mathcal{E} = \cup \{\mathcal{E}_\beta : \beta < \lambda\}$ which satisfies i) $\{\cup \mathcal{E}_\beta : \beta < \lambda\}$ paritions X, ii) for every $\beta < \lambda$, $\mathcal{E}_\beta$ is a relatively P collection of closed subsets of the subspace $X - \cup \{\cup \mathcal{E}_\mu : \mu < \beta\}$, and iii) for every $\beta < \lambda$, $\cup \{\cup \mathcal{E}_\mu : \mu < \beta\}$ is a closed set.

The collection $\mathcal{E}$ is often called a $B(P, \lambda)$-refinement of $\mathcal{U}$.

Problem. When are the properties $B(D, \lambda)$-refinable, $B(LF, \lambda)$-refinable and $B(HCP, \lambda)$-refinable equivalent? Partial answers to this question are found in [6]. We now provide a more complete answer using the following result [7].

Theorem 1 (Rudin). *Let X be a collectionwise normal space and $\mathcal{U}$ an open cover of X. If $\mathcal{U}$ has a closed hereditarily closure-perserving refinement, then $\mathcal{U}$ has a locally finite closed refinement.*

Theorem 2 *In a collectionwise normal space X, the following are equivalent.*

(i) *X is paracompact.*

(ii) *X is $B(D, \lambda)$-refinable.*

(iii) *X is $B(LF, \lambda)$-refinable.*

(iv) *X is $B(HCP, \lambda)$-refinable.*

Proof. It is known (see [6]) that (i) ≡ (ii) ≡ (iii) and clear that (iii) ⇒ (iv). Here we need only show that (iv) ⇒ (i). Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of X and $\mathcal{E} = \cup \{\mathcal{E}_\beta : \beta < \lambda\}$ a $B(HCP, \lambda)$-refinement of $\mathcal{U}$. By Theorem 3 of [6] we have that X is expandable. We construct for each $\delta < \lambda$, a LF-open partial refinement $\mathcal{W}_\delta$ of $\mathcal{U}$ such that W_δ covers $(\cup \mathcal{E}_\delta) -$

$\cup\{(\cup \mathcal{W}_\beta) : \beta < \delta\}$. To get started, $\mathcal{U}$ has a LF-closed partial refinement $\mathcal{F}_0$ which covers $\cup \mathcal{E}_0$ by Theorem 1 above. Since X is expandable, $\mathcal{U}$ has a LF-open partial refinement $\mathcal{W}_0$ covering $\cup \mathcal{E}_0$.

Now for fiixed $\delta < \lambda$, assume that for each $\beta < \delta$, there exists a LF-open partial refinement $\mathcal{W}_\beta$ of $\mathcal{U}$ such that W_β covers $[\cup \mathcal{E}_\beta] - \cup\{(\cup \mathcal{W}_\rho) : \rho < \beta\}$. Now $H = (\cup \mathcal{E}_s) - \{(\cup \mathcal{W}_\beta) : \beta < \delta\}$ is closed in X and $\mathcal{E}_\delta \mid H$ is HCP. By Theorem 1, H has a LF-closed cover $\mathcal{F}_\delta$ which partially refines $\mathcal{U}$. Thus $\mathcal{F}_\delta$ has a LF-open expansion $\mathcal{W}_\delta$ which partially refines $\mathcal{U}$, and the induction is complete. Therefore $\mathcal{U}$ has a $\sigma - LF$ open refinement and hence X is paracompact.

It is interesting to note that the $B(P, \lambda)$-refinable properties are "ordinal" dependent. In [1] a normal space is constructed for any countable ordinal λ, which is $B(D, \lambda)$-refinable but not $B(D, \beta)$-refinable for any $\beta < \lambda$. We can show the following however.

Theorem 3. *Let X be hereditarily countably metacompact. If X is $B(LF, \lambda)$-refinable, then X is $B(D, \omega \times \lambda)$-refinable.*

Proof: Let $\mathcal{U}$ be an open cover of X, and $\mathcal{E} = \cup\{\mathcal{E}_\delta : \delta < \lambda)$ a $B(LF, \lambda)$-refinement of $\mathcal{U}$. For each $n \in N$ and $\delta < \lambda$, define

$$S(\delta, n) = \{x : ord(x, \mathcal{E}_\delta) \leq n\} - \cup(\cup \mathcal{E}_\beta : \beta < \delta\}, \text{ and}$$
$$S_\delta = \{S(\delta, n) : n \in N\}.$$

Now S_δ is a countable monotone open cover of the subspace $K = X - \cup\{\cup \mathcal{E}_\beta : \beta < \delta\}$. Therefore, S_δ has a relatively closed shrink $\mathcal{F}_\delta = \{F(\delta, n) : n \in N)$ which covers the subspace K with $F(\delta, n) \subset S(\delta, n)$ for each $n \in N$.

For every $\delta < \lambda$ and $n \in N$, define

$$\mathcal{H}(\delta, n) = \{E \cap F(\delta, n) : E \in \mathcal{E}_\delta\}, \text{ and}$$
$$\mathcal{H}_\delta = \cup\{\mathcal{H}(\delta, n) : n \in N\}.$$

Since each member of $\mathcal{H}(\delta, n)$ is contained in $S(\delta, n)$, it follows that $\mathcal{H}(\delta, n)$ is a relatively n-bded-LF collection of closed subsets of the subspace K. Furthermore, $\mathcal{H}(\delta, n)$ partially refines $\mathcal{E}_\delta$, and $\cup \mathcal{H}_\delta = \cup \mathcal{E}_\delta$.

For every $\delta < \lambda$ and $n \in N$, define

$$\mathcal{K}(\delta, n) = \{H - \cup\{\cup \mathcal{H}(\delta, j) : j < n\} : H \in \mathcal{H}(\delta, n)\}, \text{ and}$$
$$\mathcal{K}_\delta = \cup\{\mathcal{K}(\delta, n) : n < N\}.$$

Define a well-order "$<$" on $\mathcal{J} = \{(\delta, n) : \delta < \lambda, n \in N\}$ such that for every (β, m), $(\delta, n) \in \mathcal{J}$.

$$(\beta, m) < (\delta, n) \text{ iff } (i)\beta < \delta, \text{ or } (ii)\beta = \delta \text{ and } m < n.$$

Let $f : \mathcal{J} \to \{\mu : \mu < \omega \times \lambda\}$ be the unique bijection which preserves this order. For each $\mu < \omega \times \lambda$, define $\mathcal{L}_\mu = \mathcal{K}(\delta, n)$ such that $f(\delta, n) = \mu$.

By construction it is easy to see that $\mathcal{L} = \cup(\mathcal{L}_\mu : \mu < \omega \times \lambda\}$ is a B(bded-LF, $\omega \times \lambda$)-refinement of $\mathcal{U}$; therefore, X is $B(D, \omega \times \lambda)$-refinable.

Open questions. (1) Can Rudin's Theorem be generalized?

(2) Can Theorems 2 and 3 be generalized?

(3) When is $B(CP, \lambda)$-refinable equivalent to the other properties?

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