62. Existence of a Rational Elliptic Surface with a Given Mordell-Weil Lattice

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In this note, we announce some results concerning the existence of a rational elliptic surface having a given structure of the Mordell–Weil lattice, which has been classified in [5]. With some arithmetic applications in mind, we consider the question over the rational number field Q. Details will be given in forthcoming papers. For general facts on Mordell–Weil lattices (MWL), we refer to [7] or [8].

1. Notation. Let $K = k(t)$ be the rational function field over an algebraically closed ground field k, and let E/K be an elliptic curve such that the associated elliptic surface (the Kodaira–Néron model)

$$f : S \to P^1$$

is a rational elliptic surface. Then the structure of the Mordell–Weil lattice $E(K)$ (by which we mean, by abuse of language, the structure of the Mordell–Weil group $E(K)$ equipped with the height pairing) is completely determined by the "trivial lattice" T formed by irreducible components of reducible singular fibres $f^{-1}(v)$ (let R be the set of such v's); T is the direct sum of simple root lattices T_v of type A, D, E and has a natural embedding into the root lattice E_8:

(1) $$T = \oplus_{v \in R} T_v \hookrightarrow E_8.$$

Namely, if we denote by $L = E(K)^0$ the narrow Mordell–Weil lattice, then L is isomorphic to the orthogonal complement of T in E_8, while $M = E(K)$ is isomorphic to the direct sum of L^* (the dual lattice of L) and a finite torsion group (see [8], Th. 10.3 or [5], Th. 3.1; we follow the latter notation here).

Further the possible structure of the triple $\{T, L, M\}$ has been classified into 74 types: No.1,...,No.74 ([5], Main Theorem).

Remark. (i) The terminology "trivial lattice" was used in [7], [8] to mean the lattice generated by T as above, the zero section and any fibre in the Néron–Severi lattice of S. The present usage is more convenient for the purpose of this note, and we hope it will cause no confusion.

(ii) We take this opportunity to correct the misprints in the table of [5]:
For No.32, $L = A_1 \oplus \langle 6 \rangle$, $M = A_1^* \oplus \langle 1/6 \rangle$ should be replaced by
$L = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$, $M = \frac{1}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. For No.70, $M = (Z/2Z)^2$ should read $M = Z/4Z$.

2. Existence theorem. The main result in this note is an existence theorem stating that all the 74 types actually occur (at least in case the ground field k has characteristic 0). More precisely, we can prove a much

more refined result: the existence of a Q-*split* example. Namely we have:

Theorem 1. *For each type* $\{T, L, M\}$ *except for the case No.68 in* [5] (*with* $T = A_2^{\oplus 4}$, $L = \{0\}$, $M = (Z/3Z)^{\oplus 2}$), *there exists an elliptic curve* E *over* $Q(t)$ *satisfying the following properties*:

$$(2) \qquad E(Q(t)) = E(k(t)) \simeq M, \quad E(Q(t))^0 = E(k(t))^0 \simeq L,$$

and the Kodaira–Néron model is a rational elliptic surface with the trivial lattice T *such that all the irreducible components of reducible fibres are* Q-*rational. In the exceptional case, the assertion holds with* Q *replaced by the field of cube root of unity, which is the smallest possible choice in view of Weil's* e_N-*pairing* (*here with* $N = 3$).

The proof is based on the theory of Mordell–Weil lattices, especially on the degeneration of Mordell–Weil lattices (cf. [5], Remark 2.7, [9]) combined with the idea of excellent families of elliptic curves (see below).

Remark. In the complex case ($k = C$), Persson [6] and Miranda [4] determined the configuration type of singular fibres of rational elliptic surfaces; in their approach, irreducible singular fibres (type I_1 or II) are taken into consideration as well as the distinction of a singular fibre with $T_v \simeq A_1$ (type I_2 or III) or with $T_v \simeq A_2$ (type I_3 or IV). (We use Kodaira's notation for singular fibres [3].) Thus the classification list is finer than that of [5] as far as singular fibres are concerned, but the structure of the Mordell–Weil lattice is not considered there. Also it seems to be a nontrivial work to write down some explicit example in each case (say by a Weierstrass equation over $C(t)$) following the indication in [6] (mostly given in terms of geometry of plane curves), and certainly this method is not suitable in most cases for constructing examples over $Q(t)$, not to mention Q-split ones in the above sense.

3. Excellent families. Of the 74 types, there are exactly 31 types $\{T, L, M\}$ for which L is a root lattice of positive rank. Such a type is called *admissible*. For an admissible type, Theorem 1 follows from Theorem 2 asserting the existence of an *excellent family* (cf. [11]). Namely we have:

Theorem 2. *For each admissible type with* L *a root lattice of rank* $r > 0$, *there exists an excellent family of elliptic curves* $\{E_\lambda\}$ *with Galois group* $W(L)$, *the Weyl group of* L. *In other words, the generic member* E_λ *is an elliptic curve defined over* $k_0(t)$ *where* $k_0 = Q(\lambda)$ ($\lambda = (p_1, \ldots, p_r)$) *is a purely transcendental extension of dimension* r *over* Q. *There exist* r *independent variables* $\{u_1, \ldots, u_r\}$ (*called splitting variables*) *in the algebraic closure* k *of* k_0 *with the following properties: first*

$$(3) \qquad \begin{cases} E_\lambda(k(t))^0 = E_\lambda(Q(u_1, \ldots, u_r)(t))^0 \simeq L \\ E_\lambda(k(t)) = E_\lambda(Q(u_1, \ldots, u_r)(t)) \simeq M \end{cases}$$

and second the Weyl group $W(L)$ *acts on the polynomial ring* $Q[u_1, \ldots, u_r]$ *in such a way that the invariant subring is*:

$$(4) \qquad Q[u_1, \ldots, u_r]^{W(L)} = Q[p_1, \ldots, p_r],$$

which means that $\{p_i\}$ *forms a set of fundamental invariants of the Weyl group* $W(L)$.

This theorem has been proven for the cases $L = E_8, E_7, E_6$ (and some

cases with $L = D_4, A_2$) in our previous work [10]; in these cases, we can take as E_λ the elliptic curve whose defining equation is given by that of the semi-universal deformation of rational double point of type L. The case $L = D_5$ has been treated by Usui [12], and about ten more cases $L = D_6, A_5, \ldots$ have been given in a recent joint work [11]. The existence of an excellent family for the remaining cases will be established in a forthcoming paper.

Theorem 1 for an admissible type is an immediate consequence of Theorem 2. For we have only to specialize the splitting variables $\{u_1, \ldots, u_r\}$ to some rational numbers $\{a_1, \ldots, a_r\}$ so that the MWL does not degenerate, which is a certain open condition (cf. [11], Th. 2 and examples).

The proof of Theorem 1 for a non-admissible type is based on the opposite idea, i.e. *degeneration of MWL*. Starting from a suitable excellent family, we choose a specialization of the splitting variables to arrange the desired set of reducible fibres, i.e. to realize a given type of the trivial lattice T. The main idea here is to consider *vanishing roots* which are analogous to *vanishing cycles* in the Milnor lattice in singularity theory (cf. [9]).

4. Application to singularity theory. If we ignore the information on the structure of MWL in Theorem 1 or 2, we can translate the results on reducible singular fibres into the corresponding results on singularities as follows :

Theorem 3. *Let T be any sublattice of E_8 which is a direct sum of root lattices of type A, D, E : $T = \oplus T_i$. Assume $T \neq A_1^{\oplus 8}, A_1^{\oplus 7}, D_4 \oplus A_1^{\oplus 4}$. Then there exists an affine surface X in the affine space $\mathbf{C}^3$ defined over $\mathbf{Q}$ which has a rational double point, say x_i, of type T_i for each i and no other singularities. Moreover each singular point x_i is a $\mathbf{Q}$-rational point of X, and the minimal resolution is obtained by blowing up only $\mathbf{Q}$-rational points so that the exceptional curves of the resolution are all defined over $\mathbf{Q}$. A single exception is the case where $T = A_2^{\oplus 4}$, in which case we need to blow up $\mathbf{Q}(\sqrt{-3})$-rational points.*

Except for the rationality statements, this result must be wellknown to singularity theorists. We do not know whether the existence in the above refined sense (i.e. the existence of a $\mathbf{Q}$-split example for every possible type) has been known.

Example. For $T = A_4 \oplus A_2 \oplus A_1$ (No.56 in [5]), consider the affine surface in (x, y, t)-space defined by the equation
$$(5) \qquad y^2 + 72xy - 10t^2y = x^3 + 60tx^2 - 15t^3x + t^5.$$
Then it has 3 singular points: A_4-singularity at $(x, y, t) = (0, 0, 0)$, A_2-singularity at $(x, y, t) = (2^8, -2^{12}, -2^5)$, and A_1-singularity at $(x, y, t) = (3^4, 3^6, -3^3)$.

To reformulate Theorem 2, note that (4) defines a Galois covering between affine spaces
$$\pi : A^r \rightarrow A^r / W(L) \simeq A^r$$
whose ramification locus is defined by $\delta(\lambda) = 0$, where the invariant δ is the square of the basic anti-invariant of the Weyl group $W(L)$ ([1], Ch.5, § 5.4, Prop.5).

Theorem 4. *For each admissible type $\{T, L, M\}$, there exists a family of*

affine surfaces $\{X_\lambda\}$ *parametrized by* $\lambda = (p_1, \ldots, p_r) \in A^r$ *such that* (i) *the generic member has only T-singularities* (i.e. *rational double points of type* T_i *as before) and* (ii) *for* $\lambda' \in A^r$, $X_{\lambda'}$ *has worse singularities than* T *if and only if* λ' *belongs to the ramification locus of* π.

These families can be used in studying the stratification of the parameter space for deformation of rational double points up to E_8, since they provide a sort of partial smoothing which "preserves" T-singularities. (Recall that the family for the case $T = \{0\}$ is the semi-universal deformation of E_8-singularity X_0.)

5. Minimal height. We mention an arithmetic consequence.

Theorem 5. *The minimal height of a non-torsion point* $P \in E(K)$ *is equal to* $1/30$ *when* E/K *runs over elliptic curves with a rational elliptic surface as the Kodaira-Néron model*:

$$\text{Min } \langle P, P \rangle = \frac{1}{30}.$$

Indeed, it follows from the classification in [5] that the said minimum is at least $1/30$, attaining this value if and only if the case No.56 with $T = A_4 \oplus A_2 \oplus A_1$, $L = \langle 30 \rangle$, $M = \langle 1/30 \rangle$ exists. Theorem 1 assures the existence, so the result. An explicit example is given by the previous equation (5), now viewed as an elliptic curve over $\boldsymbol{Q}(t)$, and its rational point $P = (t^2 + 24t, t^3 + 28t^2)$ which has norm $\langle P, P \rangle = 1/30$.

A general result in this direction has been obtained by Hindy and Silverman [2]. For instance, they give a lower bound for such a minimal height when E is an arbitrary elliptic curve over $K = k(t)$, k being of characteristic 0, such that $E(K)$ is finitely generated. (By the way, in the example in [2], p.437, one should have norm $\langle P, P \rangle = 1/14$ rather than $1/12$ as given there; it corresponds to No.47 in [5].)

It will be very interesting to determine the precise lower bound in the above-mentioned situation. A partial result such as the minimal height for elliptic $K3$ surfaces will be also interesting; this value is at least $1/120$, which is attained only if there exists an elliptic $K3$ surface with reducible fibres of types I_8, I_5, I_4, I_3, I_2 and having the Picard number 20.

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