60. *On Locally Trivial Families of Analytic Subvarieties with Locally Stable Parametrizations of Compact Complex Manifolds*

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Introduction. The purpose of this note is to outline the recent results of our study on locally trivial families, i.e., families which locally are products, of *analytic subvarieties with locally stable parametrizations* of a compact complex manifold (cf. Definition 1.2 below), parametrized by (possibly nonreduced) complex spaces. The main theorem is as follows:

Main theorem. *Let Y be a compact complex manifold. We denote by $E(Y)$ the set of all analytic subvarieties with locally stable parametrizations of Y. We denote by Z_t an analytic subvariety with a locally stable parametrization of Y corresponding to a point $t \in E(Y)$. We define a subset $\tilde{\mathfrak{X}}(Y)$ of the product space $Y \times E(Y)$ by*

$$\tilde{\mathfrak{X}}(Y) := \{(y, t) \mid t \in E(Y), y \in Z_t\}.$$

We denote by $\tilde{\pi} : \tilde{\mathfrak{X}}(Y) \to E(Y)$ the restriction of the projection map Pr_{E_Y}: $Y \times E(Y) \to E(Y)$ to $\tilde{\mathfrak{X}}(Y)$. Then $E(Y)$ and $\tilde{\mathfrak{X}}(Y)$ have the structure of Hausdorff complex spaces which enjoy the following properties:

(i) *$\tilde{\mathfrak{X}}(Y)$ is a closed complex subspace of the product complex space $Y \times E(Y)$, and $\tilde{\pi} : \tilde{\mathfrak{X}}(Y) \to E(Y)$ is a locally trivial family of analytic subvarieties with locally stable parametrizations of Y, parametrized by $E(Y)$.*

(ii) (*Universality*) *Given a locally trivial family $\pi : \mathfrak{X} \to M$ of analytic subvarieties with locally stable parametrizations of Y, parametrized by a complex space M, there exists a unique holomorphic map $f : M \to E(Y)$ such that $f^*\tilde{\mathfrak{X}}(Y) = \mathfrak{X}$.*

(iii) *We denote by $D(Y)$ the Duady space of closed complex subspaces of Y, and by $\tilde{\pi}_0 : \tilde{\mathfrak{U}}(Y) \to D(Y)$ the universal family of closed complex subspaces of Y (cf. [1]). Then the inclusion map $\iota : E(Y) \to D(Y)$ is a holomorphic immersion and $\iota^*\tilde{\mathfrak{U}}(Y) = \tilde{\mathfrak{X}}(Y)$.*

(iv) (*C^∞ triviality*) *Let $t_0 \in E(Y)$ be a point whose corresponding point of $E(Y)_{red}$ (the reduction of $E(Y)$) is non-singular, then there exist an open neighborhood N of t_0 in $E(Y)$ and a diffeomorphism $\Psi : Y \times N \to Y \times N$ over N (i.e., $pr_N \circ \Psi = pr_N$) such that $\Psi(Z_{t_0} \times N) = \tilde{\pi}^{-1}(N)$.*

(v) (*C^∞ type constancy*) *Let t and t' be two points of the same connected component of $E(Y)$, then there exists a diffeomorphism $\varphi : Y \to Y$ such that $\varphi(Z_t) = Z_{t'}$.*

These results might be considered as a generalization of Namba's results in [6] to higher dimensional singular cases. Details will be published elsewhere.

§1. Local existence of the universal locally trivial family.

1.1 Definition. A holomorphic map $f: X \to Y$ between complex manifolds is said to be *locally stable* if, for any point $q \in Y$ and any finite subset $S \subset f^{-1}(q)$, a multi-germ $f: (X, S) \to (Y, q)$ is simultaneously stable, i.e., any unfolding of f is trivial.

1.2 Definition. An analytic subvariety Z (possibly not of pure dimension) of a complex manifold Y is said to be *with a locally stable parametrization* if

(i) its normal model X is non-singular, and

(ii) the composite map $f := \iota \circ n : X \to Y$ is locally stable, where $n: X \to Z$ is the normalization map and $\iota: Z \hookrightarrow Y$ is the inclusion map.

From now on let Z be an analytic subvariety with a locally stable parametrization of a compact complex manifold Y, and let $f := \iota \circ n : X \to Y$ be the same as in Definition 1.2 unless otherwise stated. We denote by $(\boldsymbol{GC})^o$ the dual category of germs of complex spaces. We define two deformation functors D and L from $(\boldsymbol{GC})^0$ to *Set*, the category of sets, by:

$D: (M, o) \to \{$ isomorphism classes of the families of deformations of the holomorphic map $f: X \to Y$ with Y fixed, parametrized by $(M, o)\}$,

$L: (M, o) \to \{$ isomorphism classes of the locally trivial families of displacements of Z in Y, parametrized by $(M, o)\}$,

where (M, o) denotes a germ of a complex space. Given a family $(\mathcal{X}, F, \pi, M, o, \varphi)$ of deformations of the map $f: X \to Y$ with Y fixed, parametrized by (M, o), we define $\mathcal{Z} := F(\mathcal{X})$ and $\pi_1 := Pr_{M|\mathcal{Z}} : \mathcal{Z} \to M$, the restriction to $\mathcal{Z}$ of the projection map $Pr_M : Y \times M \to M$. Then, since $f: X \to Y$ is locally stable, $(\mathcal{Z}, \pi_1, M, o)$ is a locally trivial family of displacements of Z in Y, parametrized by (M, o). The correspondence $(\mathcal{X}, F, \pi, M, o, \varphi) \to (\mathcal{Z}, \pi_1, M, o)$ give a *natural transformation* between the functors D and L. Furthermore, the following Lemma 1.3 and Theorem 1.4 ensure that this correspondence is a *natural equivalence.*

1.3 Lemma. *Let $(\mathcal{X}, F, \pi, M, o, \varphi)$ be a family of deformations of the map $f: X \to Y$ with Y fixed, parametrized by a complex space. We define $\mathcal{Z} := F(\mathcal{X})$, and let $\mathcal{X} \xrightarrow{F'} \mathcal{Z} \hookrightarrow Y \times M$ be the factorization of F. Then there is an open neighborhood N of o in M such that:*

(i) $F_t : X_t \to Y \times t$ $(X_t := \pi^{-1}(t),\ F_t := F_{|X_t} : X_t \to Y = Y \times t)$ *is locally stable for any $t \in N$,* and

(ii) $F'_t : X_t \to Z_t$ $(Z_t := \mathcal{Z} \cap (Y \times t),\ F'_t := F'_{|X_t} : X_t \to Z_t)$ *is the normalization map of Z_t for any $t \in N$.*

1.4 Theorem (Relative normalization theorem). *Let $\pi_1 : \mathcal{Z} \to M$ be a locally trivial family of analytic varieties parametrized by a complex space. Then there exist a locally trivial family $\pi : \mathcal{X} \to M$ of analytic varieties parametrized by the same complex space M and a surjective holomorphic map $\nu : \mathcal{X} \to \mathcal{Z}$ over M such that $\nu_t : X_t \to Z_t$ $(X_t := \pi^{-1}(t),\ Z_t := \pi_1^{-1}(t),\ \nu_t := \nu_{|X_t} : X_t \to Z_t)$ is the normalization of Z_t for any $t \in M$. Furthermore, the family $\pi : \mathcal{X} \to M$ and the surjective holomorphic map $\nu : \mathcal{X} \to \mathcal{Z}$ over M are uniquely determined up to* *biholomorphic maps over* M.

Proof. By the local triviality of the family, for each point $p \in \mathcal{Z}$, there exist open neighborhoods $\mathcal{U}$ of p in $\mathcal{Z}$, V of $\pi_1(p)$ in M and a biholomorphic map $\varphi : \mathcal{U} \to U \times V$, where $U := \mathcal{U} \cap \pi_1^{-1}(\pi_1(p))$, such that $\pi_{1|\mathcal{U}} = Pr_V \circ \varphi$. We may assume that V is a closed complex subspace of a domain D in a complex number space C^n. We denote by $\tilde{\mathcal{O}}_{U\times D}$ the sheaf of germs of *weakly holomorphic functions* on $U \times D$ (i.e., holomorphic functions on $(U \setminus S(U)) \times D$, which are locally bounded on $U \times D$, where $S(U)$ denotes the singular locus of U), which is a coherent $\mathcal{O}_{U\times D}$-module (cf. [7]). We define

$$\tilde{\mathcal{O}}_{\mathcal{U}/V} := (\varphi^{-1})_* \{\mathcal{O}_{U\times V} \otimes_{\mathcal{O}_{U\times D}} [\tilde{\mathcal{O}}_{U\times D}/(Pr_D^* \mathcal{I}_V) \cdot \tilde{\mathcal{O}}_{U\times D}]\},$$

where $\mathcal{I}_V$ denotes the ideal sheaf of V in $\mathcal{O}_D$. We can patch up these sheaves defined locally and obtain a global $\mathcal{O}_{\mathcal{Z}}$-module, which we denote by $\tilde{\mathcal{O}}_{\mathcal{Z}/M}$ and call the *sheaf of germs of weakly holomorphic functions along fibers of a locally trivial family* $\pi_1 : \mathcal{Z} \to M$. From the definition it follows that $\tilde{\mathcal{O}}_{\mathcal{Z}/M}$ is a coherent $\mathcal{O}_{\mathcal{Z}}$-module. Let $\nu : \mathcal{X} \to \mathcal{Z}$ be the analytic spectrum of $\tilde{\mathcal{O}}_{\mathcal{Z}/M}$ (cf. [5]) and we define $\pi := \pi_1 \circ \nu : \mathcal{X} \to M$. Then the surjective holomorphic map $\nu : \mathcal{X} \to \mathcal{Z}$ over M and the family $\pi : \mathcal{X} \to M$ enjoy the properties in the theorem. Q.E.D.

1.5 Corollary. *For an analytic subvariety* Z *with a locally stable parametrization of a compact complex manifold* Y, *there exists a family* $(\bar{\mathcal{Z}}, \bar{\pi}_1, \bar{M}, \bar{o})$ *of locally trivial displacements of* Z *in* Y, *parametrized by a complex space such that*:

(i) $\bar{\pi}_1^{-1}(t)$ *is an analytic subvariety with a locally stable parametrization of* Y *for any* $t \in \bar{M}$,

(ii) $\bar{\pi}_1^{-1}(t) \neq \bar{\pi}_1^{-1}(t')$ *for any* $t, t' \in \bar{M}$ *with* $t \neq t'$, *and*

(iii) *universal at any point* $t \in \bar{M}$.

Proof. Let $n : X \to Z$ be the normalization of Z and $\iota : Z \hookrightarrow Y$ the inclusion map. We define $f := \iota \circ n : X \to Y$. By Flenner's theorem [2, Theorem (8.5)] there exists a semi-universal family $(\bar{\mathcal{X}}, \bar{F}, \bar{\pi}, \bar{M}, \bar{o}, \bar{\varphi})$ of deformations of $f : X \to Y$ with Y fixed. We define $\bar{\mathcal{Z}} := \bar{F}(\bar{\mathcal{X}})$ and $\bar{\pi}_1 := Pr_{\bar{M}|\bar{\mathcal{Z}}} : \bar{\mathcal{Z}} \to \bar{M}$, the restriction to $\bar{\mathcal{Z}}$ of the projection map $pr_{\bar{M}} : Y \times \bar{M} \to \bar{M}$. By Lemma 1.3, shrinking $\bar{M}$ around $\bar{o}$ if necessary, we may assume that the family $(\bar{\mathcal{Z}}, \bar{\pi}_1, \bar{M}, \bar{o})$ satisfies the condition (i) in the corollary. It is the *Kuranishi family* for locally trivial displacements of Z in Y, because the functors D and L are equivalent. Let $D(Y)$ be the Duady space of closed complex subspaces of Y, $\tilde{\pi}_o : \tilde{\mathcal{U}}(Y) \to D(Y)$ the universal family of closed complex subspaces of Y, and $\tilde{o}$ the point of $D(Y)$ corresponding to the analytic subvariety Z. By (0.2) Corollary in [3], for any point $z \in Z$, there exists a locally closed complex subspace N_z of $D(Y)$ containing the point $\tilde{o}$, which enjoys the following property:

If $\alpha : (T, o) \to (D(Y), \tilde{o})$ *is a holomorphic map between germs of complex spaces, then the induced family* $(\alpha^* \tilde{\mathcal{U}}(Y), z) \to (T, o)$ *of deformations of the germ* (Z, z) *of a complex space is isomorphic to the trivial deformation* $(Z, z) \times (T, o) \to (T, o)$ *if, and only if,* α *factorizes over* $(N_z, \tilde{o})$.

We define $\tilde{N} := \cap_{z\in Z} N_z$ (the intersection as complex subspaces), $\tilde{\mathcal{Z}} := \tilde{\mathcal{U}}(Y)_{|\tilde{N}}$ (the restriction of $\tilde{\mathcal{U}}(Y)$ over $\tilde{N}$), and $\tilde{\pi} := \tilde{\pi}_{o|\tilde{\mathcal{Z}}} : \tilde{\mathcal{Z}} \to \tilde{N}$ (the restriction of $\tilde{\pi}_0 : \tilde{\mathscr{U}}(Y) \to D(Y)$ to $\tilde{\mathscr{X}}$. Then, by the definition of the family $(\tilde{\mathscr{X}}, \tilde{\pi}, \tilde{N}, \tilde{o})$ it is a locally trivial family of displacements of Z in Y that satisfies the condition (ii) in the corollary and is *maximal* (= versal) at $\tilde{o}$. Comparing $(\bar{\mathscr{X}}, \bar{\pi}_1, \bar{M}, \bar{o})$ with $(\tilde{\mathscr{X}}, \tilde{\pi}, \tilde{N}, \tilde{o})$, we conclude that they are isomorphic to each other in sufficiently small open neighborhoods of $\bar{o}$ and $\tilde{o}$ in $\bar{M}$ and $\tilde{M}$, respectively. From this fact it follows that $(\bar{\mathscr{X}}, \bar{\pi}_1, \bar{M}, \bar{o})$ satisfies the conditions (ii) and (iii) in the corollary in a sufficiently small open neighborhood of $\bar{o}$ in $\bar{M}$. Q.E.D.

§2. Proof of the main theorem. For any analytic subvariety Z of Y, the family $(\bar{\mathscr{X}}, \bar{\pi}_1, \bar{M}, \bar{o},)$ in Corollary 1.5 gives a structure of a locally trivial family of analytic subvarieties with locally stable parametrizations of Y to $\tilde{\pi} : \tilde{\mathscr{X}}(Y) \to E(Y)$ around Z. By the uniqueness of the universal family $(\bar{\mathscr{X}}, \bar{\pi}_1, \bar{M}, \bar{o},)$ up to isomorphisms, these local structures patch up to give a global structure of locally trivial family of analytic subvarieties with locally stable parametrizations of Y to $\tilde{\pi} : \tilde{\mathscr{X}}(Y) \to E(Y)$. Following Namba's proof in [6], we can prove that $E(Y)$ is Hausdorff. The assertions (i)-(iii) in Main theorem follows directly from the above construction. The assertions (iv) and (v) follows from the C^∞ triviality of deformations of a locally stable holomorphic map (cf. [9]).

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