33. Notes on Some Classical Series Associated with Discrete Subgroups of $U(1,n;C)$ on $\partial B^n\times\partial B^n\times\partial B^n$

By Shigeyasu KAMIYA
Department of Mechanical Engineering, Okayama University of Science

(Communicated by Shokichi IYANAGA, M. J. A., June 9, 1992)

Let $U(1,n;\boldsymbol{C})$ be the group of unitary transformations. In the previous paper [2], we discussed the action of discrete subgroups of $U(1,n;\boldsymbol{C})$ on $\partial B^n\times\partial B^n\times\cdots\times\partial B^n$, where ∂B^n is the boundary of the complex unit ball. In [4], P. J. Nicholls considered the convergence of some series associated with discrete subgroups of Möbius transformations on the products of the boundary of the unit ball in real n-space.

Our purpose is to show two theorems on some classical series associated with discrete subgroups of $U(1,n;\boldsymbol{C})$ acting on $\partial B^n\times\partial B^n\times\partial B^n$. Throughout this paper G denotes a discrete subgroup of $U(1,n;\boldsymbol{C})$. Let $\{g_1,g_2,\cdots\}$ be a complete list of elements of G. If g_k is an element of G, then g_k is represented by a matrix $(a_{ij}^{(k)})_{1\le i,j\le n+1}$. Let $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)$ and $z=(z_1,\cdots,z_n)$ be points in ∂B^n.

Theorem 1. *The series*

$$\sum_{g_k\in G}\left(\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}x_{j-1}\right|\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}y_{j-1}\right|\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}z_{j-1}\right|\right)^{-2n}$$

converges for almost every triple (x,y,z) in $\partial B^n\times\partial B^n\times\partial B^n$.

Theorem 2. *If $\sum_{g_k\in G}|a_{11}^{(k)}|^{-m}$ converges for $m>0$, then the series*

$$\sum_{g_k\in G}\left(\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}x_{j-1}\right|\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}y_{j-1}\right|\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}z_{j-1}\right|\right)^{-m}$$

converges for every distinct points x, y and z in ∂B^n.

We shall give our proofs.

Proof of Theorem 1. Let $\Gamma(g_k)$ be the set of (x,y,z) in $\partial B^n\times\partial B^n\times\partial B^n$ for which

$$\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}x_{j-1}\right|\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}y_{j-1}\right|\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}z_{j-1}\right|>1.$$

Set

$$F=\bigcap_{g_k\neq id}\Gamma(g_k).$$

It follows from [2, Theorem 11] that F is a fundamental set for the group action on $\partial B^n\times\partial B^n\times\partial B^n$. Since F is of positive measure and has no G-equivalent points,

$$\sum_{g_k\in G}\sigma^*(g_k(F))<\infty,$$

where σ^* is the product measure on $\partial B^n\times\partial B^n\times\partial B^n$ derived from the measure σ on ∂B^n (see [2, p. 288]). For $(x,y,z)\in F$

$$\sum_{g_k\in G}\sigma^*(g_k(F))=\sum_{g_k\in G}\int_{g_k(F)}d\sigma^*$$

$$=\int_F \sum_{g_k\in G}\left(\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}x_{j-1}\right|\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}y_{j-1}\right|\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}z_{j-1}\right|\right)^{-2n}d\sigma(x)d\sigma(y)d\sigma(z).$$

Hence the series

$$\sum_{g_k\in G}\left(\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}x_{j-1}\right|\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}y_{j-1}\right|\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}z_{j-1}\right|\right)^{-2n}$$

converges almost everywhere in F. Thus our proof is complete.

Let $s=(s_1,\cdots,s_n)$ and $t=(t_1,\cdots,t_n)$ be points of $\overline{B^n}$. We define

$$d^*(s,t)=\left|1-\sum_{j=1}^{n}\overline{s_j}t_j\right|^{1/2}$$

(see [2, p. 288] and [3, Proposition 3.2]).

To prove Theorem 2 we prepare two lemmas.

Lemma 3. *Let p_k be a point such that $g_k(p_k)=0$. Define*

$$\delta=\min\{d^*(x,y),\ d^*(y,z),\ d^*(z,x)\},$$

where x, y and z are distinct points in ∂B^n. Then at least one of $d^*(p_k,x)$, $d^*(p_k,y)$ and $d^*(p_k,z)$ is greater than $\delta/2$.

Proof. Suppose that all three are smaller than $\delta/2$. Then we have

$$d^*(x,y)\le d^*(p_k,x)+d^*(p_k,y)<\delta,$$
$$d^*(y,z)\le d^*(p_k,y)+d^*(p_k,z)<\delta,$$
$$d^*(z,x)\le d^*(p_k,z)+d^*(p_k,x)<\delta.$$

This is a contradiction.

Lemma 4. *Let g_k be an element of G with $g_k(p_k)=0$. Then*

$$\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}y_{j-1}\right|^{-1}\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}z_{j-1}\right|^{-1}\le 2d^*(y,z)^{-2}.$$

Proof. First we note that $p_k=(\overline{-a_{12}^{(k)}/a_{11}^{(k)}},\ \overline{-a_{13}^{(k)}/a_{11}^{(k)}},\ \cdots,\ \overline{-a_{1,n+1}^{(k)}/a_{11}^{(k)}})$ and $d^*(g_k(y),g_k(z))\le\sqrt{2}$. Using [2, Lemma 5], we see

$$\begin{aligned}\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}y_{j-1}\right|^{-1}\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}z_{j-1}\right|^{-1}&=\frac{d^*(p_k,p_k)^2}{d^*(p_k,y)^2d^*(p_k,z)^2}\\&=\frac{d^*(g_k(y),g_k(z))^2}{d^*(y,z)^2}\\&\le 2d^*(y,z)^{-2}.\end{aligned}$$

Thus our lemma is proved.

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Using Lemma 3, we may assume that $d^*(p_k,x)>\delta/2$. Then we see

$$d^*(p_k,x)^2=\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}x_{j-1}\right||a_{11}^{(k)}|^{-1}>\delta^2/4.$$

Therefore $|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}x_{j-1}|^{-1}\le 4|a_{11}^{(k)}|^{-1}\delta^{-2}$. It follows from Lemma 4 that

$$\begin{aligned}&\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}x_{j-1}\right|^{-1}\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}y_{j-1}\right|^{-1}\left|a_{11}^{(k)}+\sum_{j=2}^{n+1}a_{1j}^{(k)}z_{j-1}\right|^{-1}\\&\qquad\le 8|a_{11}^{(k)}|^{-1}\delta^{-2}d^*(y,z)^{-2}\\&\qquad\le 8|a_{11}^{(k)}|^{-1}\delta^{-4}.\end{aligned}$$

Thus our theorem is completely proved.

Remark 5. It is known that if $m>2n$, then the series $\sum_{g_k\in G}|a_{11}^{(k)}|^{-m}$ converges for a discrete subgroup G of $U(1,n;\boldsymbol{C})$ (see [1, Theorem 5.2]).

Remark 6. In the case where G acts on $\partial B^n \times \partial B^n \times \cdots \times \partial B^n$ with more than three factors, similar results are proved by a slight modification of our proofs.

References

[1] S. Kamiya: Discrete subgroups of convergence or divergence type of $U(1, n; \boldsymbol{C})$. Math J. Okayama Univ., **26**, 179–191 (1984).

[2] ——: Discrete subgroups of $U(1, n; \boldsymbol{C})$ on the product space $\partial B^n \times \partial B^n \times \cdots \times \partial B^n$. Quart. J. Math. Oxford, (2) **41**, 287–294 (1990).

[3] ——: Discrete subgroups of convergence type of $(1, n; \boldsymbol{C})$. Hiroshima Math. J., **21**, 1–21 (1991).

[4] P. J. Nicholls: The ergodic theory of discrete groups. London Math. Soc., Lecture Note Series, vol. 143 (Cambridge Univ. Press 1989).