32. *Some Problems of Diophantine Approximation in the Theory of the Riemann Zeta Function*

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§ **Introduction.** Let α be a positive number. The distribution of the fractional part $\{\alpha n\}$ of αn has been studied extensively. It is well-known that it depends heavily on the arithmetic nature of α. We may briefly recall this fact for a quadratic irrational α as follows. It was shown by Hardy-Littlewood [6] and Ostrowski [8] that

$$\sum_{n\le X}\left(\{\alpha n\}-\frac{1}{2}\right)\ll \log X.$$

Hecke [7] has shown, in fact, that if α is $\sqrt{D}$ or $1/\sqrt{D}$ with a positive square free integer $D\equiv 2$ or 3 (mod 4), then for any $\varepsilon>0$

$$\sum_{n\le X}\left(\{\alpha n\}-\frac{1}{2}\right)\log^2\frac{X}{n}=A_1\log^3 X+A_2\log^2 X+A_3\log X$$
$$+\sum_{m=-\infty}^{+\infty}C_m X^{(2\pi i m)/(\log\eta_D)}+O(X^{-1+\varepsilon}),$$

where A_1, A_2, A_3 and C_m are some constants, $C_m=O(|m|^{-2+\varepsilon})$ for $m\neq 0$ and η_D is the fundamental unit of the quadratic number field $\boldsymbol{Q}(\sqrt{D})$ or the square of it. The author [4] [5] has extended his result and shown that for any $\varepsilon>0$

$$\sum_{n\le X}\left(\{\alpha n\}-\frac{1}{2}\right)\log\frac{X}{n}=\frac{1}{2}G_1(\alpha)\log^2 X+G_2(\alpha)\log X$$
$$+\sum_{m=-\infty}^{+\infty}C'_m X^{(2\pi i m)/(\log\eta_D)}+O(X^{-(1/3)+\varepsilon}),$$

where $G_1(\alpha)$ and $G_2(\alpha)$ can be explicitly written down in terms of the continued fraction expansion of α and $C'_m=O(|m|^{-(4/3)+\varepsilon})$ for $m\neq 0$.

Here we are concerned with the distribution of

$$\left\{\alpha\frac{\gamma}{2\pi}\right\}-\frac{1}{2},$$

where γ runs over the positive imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$. Our main problem is to find an asymptotic formula for the sum

$$\sum_{\gamma\le T}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}-\frac{1}{2}\right)$$

and determine how it depends on α. Our result is not precise enough for

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this sum. However, we shall give a finer result on the asymptotic behavior of the sum

$$\sum_{\gamma\le T}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}^2-\left\{\alpha\frac{\gamma}{2\pi}\right\}+\frac{1}{6}\right)$$

and see, in particular, a singular property when e^α is a prime power.

The following theorems will be proved. Let $N(T)$ denote the number of the zeros of $\zeta(s)$ in $0<\Im s<T$, which is known to be

$$\sim\frac{T}{2\pi}\log T.$$

Let R. H. be the abbreviation of the Riemann Hypothesis.

Theorem 1. *For any positive α and $T>T_0$, we have*

$$\frac{1}{N(T)}\sum_{\gamma\le T}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}-\frac{1}{2}\right)\ll\sqrt{\frac{\log\log T}{\log T}}.$$

Theorem 2 (Under R. H.). *For any positive α, positive ε and $T>T_0$, we have*

$$\frac{1}{N(T)}\sum_{\gamma\le T}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}-\frac{1}{2}\right)\ll\frac{1}{(\log T)^{1-\varepsilon}}.$$

Theorem 3. *For any positive α and $T>T_0$, we have*

$$\frac{1}{N(T)}\sum_{\gamma\le T}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}^2-\left\{\alpha\frac{\gamma}{2\pi}\right\}+\frac{1}{6}\right)\ll\frac{\log\log T}{\log T}.$$

Theorem 4 (Under R. H.). *Suppose that either α or e^α is algebraic. Then for any positive ε and $T>T_0$, we have*

$$\sum_{\gamma\le T}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}^2-\left\{\alpha\frac{\gamma}{2\pi}\right\}+\frac{1}{6}\right)=-\frac{T}{2\pi^3}\frac{\Lambda(e^{G\alpha})}{G^2}Li_2(e^{-(G/2)\alpha})+O\left(\frac{T}{(\log T)^{1-\varepsilon}}\right),$$

where $\Lambda(x)=\log p$ if $x=p^k$ with a prime number p and an integer $k\ge 1$, $=0$ otherwise, we put

$$Li_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}$$

and G is either the minimum integer $n(\ge 1)$ such that $e^{n\alpha}$ is a prime power, or $1/\alpha$ if such n does not exist.

It is clear from the proof of Theorem 4 that the same conclusion holds for α of the form $\beta_0+\sum_{j=1}^{M}\beta_j\log\alpha_j$ with non-zero algebraic numbers α_j, $j=1,2,3,\cdots,M$ and algebraic numbers β_j, $j=0,1,2,\cdots,M$. Other cases for α are included in the following general theorem.

Theorem 5 (Under R. H.). *For any positive α and $T>T_0$, we have*

$$\frac{1}{N(T)}\sum_{\gamma\le T}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}^2-\left\{\alpha\frac{\gamma}{2\pi}\right\}+\frac{1}{6}\right)\ll\frac{1}{\log T}.$$

As we know, $\{x\}-1/2=B_1(\{x\})$ and $\{x\}^2-\{x\}+1/6=B_2(\{x\})$, with the Bernoulli polynomials B_1 and B_2. Similarly, we can evaluate the sums

$$\sum_{\gamma\le T}B_n\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}\right)$$

for $n\ge 3$.

§2. Proof of Theorem 1. Let H be a sufficiently large number which well be chosen later. We decompose our sum as follows.

$$S\equiv\sum_{\gamma\le T}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}-\frac{1}{2}\right)$$

$$=\sum_{\substack{\gamma\le T\\ 1/H\le\{\alpha(\gamma/2\pi)\}\le 1-1/H}}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}-\frac{1}{2}\right)+\sum_{\substack{\gamma\le T,0\le\{\alpha(\gamma/2\pi)\}<1/H\\ 1-1/H<\{\alpha(\gamma/2\pi)\}<1}}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}-\frac{1}{2}\right)$$

$$=S_1+S_2,\quad\text{say.}$$

To estimate S_2, we shall use the following lemma which gives a discrepancy estimate of the uniform distribution of $\alpha(\gamma/2\pi)$.

Lemma 1. *For any $\alpha>0$, any positive ε and for $T>T_0$, we have*

$$\frac{1}{N(T)}\left|\left\{\gamma\le T\,;\,0\le\left\{\alpha\frac{\gamma}{2\pi}\right\}\le\beta\right\}\right|=\beta+O\left(\frac{1}{(\log T)^{1-\varepsilon}}\right)$$

uniformly for β in $0\le\beta\le 1$.

This is proved in Fujii [2].

Applying this we get

$$S_2\ll\sum_{\substack{\gamma\le T\\ 0\le\{\alpha(\gamma/2\pi)\}<1/H}}\cdot 1+\sum_{\substack{\gamma\le T\\ 1-1/H<\{\alpha(\gamma/2\pi)\}<1}}\cdot 1\ll\frac{1}{H}N(T)+N(T)(\log T)^{-1+\varepsilon}.$$

For S_1, we notice first that if $\alpha(\gamma/2\pi)$ is not an integer,

$$\left\{\alpha\frac{\gamma}{2\pi}\right\}-\frac{1}{2}=-\sum_{1\le k\le H}\frac{1}{k\pi}\sin(k\alpha\gamma)+O\left(\frac{1}{H\|\alpha(\gamma/2\pi)\|}\right),$$

where $\|x\|$ denotes the distance of x from a nearest integer.

Using this expression, we get

$$S_1=-\sum_{1\le k\le H}\frac{1}{k\pi}\sum_{\substack{\gamma\le T\\ 1/H\le\{\alpha(\gamma/2\pi)\}\le 1-1/H}}\sin(k\alpha\gamma)+O\left(\frac{1}{H}\sum_{\substack{\gamma\le T\\ 1/H\le\{\alpha(\gamma/2\pi)\}\le 1-1/H}}\frac{1}{\|\alpha(\gamma/2\pi)\|}\right)$$

$$=S_3+S_4,\quad\text{say}$$

$$S_4\ll\frac{1}{H}\sum_{\substack{\gamma\le T\\ 1/H\le\{\alpha(\gamma/2\pi)\}\le 1/2}}\frac{1}{\{\alpha(\gamma/2\pi)\}}+\frac{1}{H}\sum_{\substack{\gamma\le T\\ 1/2<\{\alpha(\gamma/2\pi)\}\le 1-1/H}}\frac{1}{1-\{\alpha(\gamma/2\pi)\}}$$

$$\ll\frac{1}{H}\sum_{1\le m\le H/2}\sum_{\substack{\gamma\le T\\ m/H\le\{\alpha(\gamma/2\pi)\}<(m+1)/H}}\frac{1}{\{\alpha(\gamma/2\pi)\}}$$

$$+\frac{1}{H}\sum_{1\le m\le H/2}\sum_{\substack{\gamma\le T\\ m/H\le 1-\{\alpha(\gamma/2\pi)\}<(m+1)/H}}\frac{1}{1-\{\alpha(\gamma/2\pi)\}}$$

$$\ll\sum_{1\le m\le H/2}\frac{1}{m}\sum_{\substack{\gamma\le T\\ m/H\le\{\alpha(\gamma/2\pi)\}<(m+1)/H}}\cdot 1+\sum_{1\le m\le H/2}\frac{1}{m}\sum_{\substack{\gamma\le T\\ 1-(m+1)/H<\{\alpha(\gamma/2\pi)\}\le 1-m/H}}\cdot 1.$$

Using Lemma 1 again, this is

$$\ll\log H\left(\frac{N(T)}{H}+N(T)(\log T)^{-1+\varepsilon}\right).$$

We turn to estimate S_3.

$$S_3=-\sum_{1\le k\le H}\frac{1}{k\pi}\sum_{\gamma\le T}\sin(k\alpha\gamma)+\sum_{1\le k\le H}\frac{1}{k\pi}\sum_{\substack{\gamma\le T,0\le\{\alpha(\gamma/2\pi)\}<1/H\\ 1-1/H<\{\alpha(\gamma/2n)\}<1}}\sin(k\alpha\gamma)$$

$$=S_5+S_6,\quad\text{say.}$$

Using Lemma 1, we get as before

$$S_6 \ll \log H\Big(\frac{N(T)}{H} + N(T)(\log T)^{-1+\varepsilon}\Big).$$

Finally, we shall estimate S_5. For this purpose we shall use the following lemma which has been proved in [3].

Lemma 2. *For* $1 < X \ll T^{8/7-\varepsilon}$, $\varepsilon > 0$ *and* $T > T_0$,

$$\sum_{\gamma \le T} X^{i\gamma} \ll T \log X + \mathrm{Min}\Big(\frac{\log T}{\log X}, \log T\Big).$$

Using this we get

$$S_5 \ll \sum_{1 \le k \le H} \frac{1}{k}\Big(Tk + \mathrm{Min}\Big(\frac{\log T}{k}, T \log T\Big)\Big) \ll TH.$$

Consequently, we get

$$S \ll \log H\Big(\frac{N(T)}{H} + N(T)(\log T)^{-1+\varepsilon}\Big) + TH.$$

Choosing $H = \sqrt{\log T \log\log T}$, we get

$$\frac{S}{N(T)} \ll \sqrt{\frac{\log\log T}{\log T}}.$$

§ 3. Proof of Theorem 2. If we assume R. H., then we can use an improvement of Lemma 2 in the following form (cf. Fujii [3] for a more precise result).

Lemma 3 (Under R. H.). *For* $X > 1$ *and* $T > T_0$, *we have*

$$\sum_{\gamma \le T} X^{i\gamma} = -\frac{T}{2\pi}\frac{\Lambda(X)}{\sqrt{X}} + O\Big(\frac{\log T}{\log X} + \sqrt{X}\log X \frac{\log T}{(\log\log T)^2}$$
$$+ \sqrt{X}\log(3X)\log\log(3X) + \frac{\log(2X)}{\sqrt{X}}\mathrm{Min}\Big(T, \frac{1}{|\log X/P(X)|}\Big)\Big),$$

where $P(X)$ *is the nearest prime power other than* X *itself.*

If we apply this, then S_5 in the previous section is

$$\ll \sum_{1 \le k \le H} \frac{1}{k}\Big(\frac{\log T}{k} + e^{k\alpha/2} k \log(3k) + e^{k\alpha/2} k \frac{\log T}{(\log\log T)^2} + \frac{k}{e^{k\alpha/2}} T\Big).$$

Here we choose $H = C \log T$ with a sufficiently small positive C. Then this is

$$\ll T.$$

Consequently, we get

$$\sum_{\gamma \le T}\Big(\Big\{\alpha\frac{\gamma}{2\pi}\Big\} - \frac{1}{2}\Big) \ll \log H\Big(\frac{N(T)}{H} + N(T)(\log T)^{-1+\varepsilon}\Big) + T$$
$$\ll N(T)(\log T)^{-1+\varepsilon}.$$

§ 4. Proof of Theorems 3, 4 and 5. We shall prove Theorems 4 and 5 first. By the Fourier expansion of $\{x\}^2 - \{x\} + 1/6$, we get

$$\sum_{\gamma \le T}\Big(\Big\{\alpha\frac{\gamma}{2\pi}\Big\}^2 - \Big\{\alpha\frac{\gamma}{2\pi}\Big\} + \frac{1}{6}\Big)$$
$$= \sum_{\gamma \le T}\sum_{1 \le n \le H} \frac{1}{\pi^2 n^2}\cos(n\alpha\gamma) + O\Big(\frac{1}{H}\sum_{\gamma \le T}\mathrm{Min}\Big(1, \frac{1}{H\|\alpha(\gamma/2\pi)\|}\Big)\Big)$$
$$= U_1 + U_2, \quad \text{say},$$

where we suppose that $1 \ll H \le C \log T$ with a sufficiently small positive number C.

Using Lemma 1 as in the previous section, we get

$$U_2 \ll \frac{1}{H} \sum_{\substack{\gamma \le T, 0 \le \{\alpha(\gamma/2\pi)\} < 1/H \\ 1-1/H < \{\alpha(\gamma/2\pi)\} < 1}} \cdot 1 + \frac{1}{H^2} \sum_{1 \le m \le H/2} \sum_{\substack{\gamma \le T \\ m/H \le \{\alpha(\gamma/2\pi)\} < (m+1)/H}} \frac{1}{\{\alpha(\gamma/2\pi)\}}$$
$$+ \frac{1}{H^2} \sum_{1 \le m \le H/2} \sum_{\substack{\gamma \le T \\ m/H \le 1 - \{\alpha(\gamma/2\pi)\} < (m+1)/H}} \frac{1}{1-\{\alpha(\gamma/2\pi)\}}$$
$$\ll \log H \left(\frac{N(T)}{H^2} + N(T) \frac{(\log T)^{-1+\varepsilon}}{H} \right).$$

Using Lemma 3, we get

$$U_1 = \sum_{1 \le n \le H} \frac{1}{\pi^2 n^2} \sum_{\gamma \le T} \cos(n\alpha\gamma)$$
$$= \sum_{1 \le n \le H} \frac{1}{\pi^2 n^2} \left(-\frac{T}{2\pi} \frac{\Lambda(e^{n\alpha})}{\sqrt{e^{n\alpha}}} + O\left(\frac{\log T}{n}\right) + O\left(e^{n\alpha} n \frac{\log T}{(\log\log T)^2}\right) \right.$$
$$\left. + O(e^{n\alpha/2} n \log(3n)) + O\left(\frac{n}{e^{n\alpha/2}} \operatorname{Min}\left(T, \frac{1}{|\log e^{n\alpha}/(P(e^{n\alpha}))|}\right)\right)\right)$$
$$= -\frac{T}{2\pi^3} \sum_{n=1}^{\infty} \frac{\Lambda(e^{n\alpha})}{n^2 e^{n\alpha/2}} + O(T^{\varepsilon}) + O\left(\sum_{1 \le n \le H} \frac{1}{n e^{n\alpha/2}} \operatorname{Min}\left(T, \frac{1}{|\log e^{n\alpha}/(P(e^{n\alpha}))|}\right)\right).$$

Suppose first that e^{α} is algebraic. Then by the formula of 1.7 in p. 3 of Baker [1], we get for $n \ge 1$ and with some positive constant D depending only on α,

$$\left| \log \frac{e^{n\alpha}}{P(e^{n\alpha})} \right| = |n \log e^{\alpha} - \log P(e^{n\alpha})| \ge e^{-Dn}.$$

Consequently, the last remainder term is

$$\ll \sum_{1 \le n \le H} \frac{e^{Dn}}{e^{n\alpha/2} n} \ll T^{\varepsilon}.$$

Choosing $H = C \log T$, we get in this case

$$\sum_{\gamma \le T} \left(\left\{ \alpha \frac{\gamma}{2\pi} \right\}^2 - \left\{ \alpha \frac{\gamma}{2\pi} \right\} + \frac{1}{6} \right) = -\frac{T}{2\pi^3} \frac{\Lambda(e^{G\alpha})}{G^2} Li_2(e^{-(G/2)\alpha}) + O\left(\frac{T}{(\log T)^{1-\varepsilon}}\right).$$

Suppose next that α is algebraic. Then by Theorem in p. 1 of Baker [1], we get for $n \ge 1$ and with some positive constant D' depending only on α,

$$\left| \log \frac{e^{n\alpha}}{P(e^{n\alpha})} \right| = |n\alpha - \log P(e^{n\alpha})| \ge e^{-D'n \log(3n)}.$$

Then, choosing $H = C(\log T)/(\log\log T)$, the last remainder term in U_1 is seen to be

$$\ll T^{\varepsilon}.$$

Hence in this case we have also the same evaluation as the first case. Thus Theorem 4 is proved.

Generally, using a trivial estimate

$$\sum_{1 \le n \le H} \frac{1}{n e^{n\alpha/2}} \operatorname{Min}\left(T, \frac{1}{|\log e^{n\alpha}/(P(e^{n\alpha}))|}\right) \ll T,$$

we get

$$\sum_{\gamma\le T}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}^2-\left\{\alpha\frac{\gamma}{2\pi}\right\}+\frac{1}{6}\right)\ll T.$$

This is our Theorem 5.

To prove Theorem 3, we use the same argument as above with $H=\sqrt{\log T}$ except the treatment of U_1. For U_1, we use Lemma 2 and get

$$U_1\ll\sum_{1\le n\le H}\frac{1}{n^2}\left(Tn+\frac{\log T}{n}\right)\ll T\log H.$$

Thus we get

$$\sum_{\gamma\le T}\left(\left\{\alpha\frac{\gamma}{2\pi}\right\}^2-\left\{\alpha\frac{\gamma}{2\pi}\right\}+\frac{1}{6}\right)\ll T\log H+\left(\frac{T\log T}{H^2}+\frac{T\log T}{H(\log T)^{1-\varepsilon}}\right)\log H$$
$$\ll T\log\log T.$$

This is our Theorem 3.

§ 5. Concluding remarks. The present method can be applied to estimate the sum

$$\sum_{p\le X}\left(\{\alpha p\}-\frac{1}{2}\right),$$

where p runs over the prime numbers. The corresponding lemmas are supplied by Vaughan in Theorems 1 and 2 of [10].

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