22. *Examples of Essentially Non-Banach Representations**)

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Let G be a locally compact unimodular group, K a compact subgroup of G. Let $\{\mathfrak{H}, T(x)\}$ be a topologically irreducible representation of G on a locally convex complete Hausdorff topological vector space $\mathfrak{H}$. We assume there exists an equivalence class δ of irreducible representation of K which is contained finitely many times in $\{\mathfrak{H}, T(x)\}$. Then the subspace $\mathfrak{H}(\delta)$ of all vectors transformed according to δ under $T(k)$, $k \in K$, is finite-dimensional, and there exists a usual projection $E(\delta)$ of $\mathfrak{H}$ onto $\mathfrak{H}(\delta)$. After R. Godement [1] we call the function $\phi_\delta(x) = \text{trace}\,[E(\delta)T(x)]$, $x \in G$, a *spherical function* of type δ. A function $\rho(x)$ on G is called a *seminorm* if it is positive-valued, lower semicontinuous and satisfies $\rho(xy) \leqq \rho(x)\rho(y)$ for $x, y \in G$. If there exists a seminorm $\rho(x)$ such that $|\phi_\delta(x)| \leqq \rho(x)$ for $x \in G$, then ϕ_δ is called *quasi-bounded*. In the case when $\mathfrak{H}$ is a Banach space, the corresponding spherical function ϕ_δ is quasi-bounded.

Even if a spherical function is defined from a non-Banach representation, it can be quasi-bounded, or equivalently equal to the one which is obtained from a Banach representation. For example, in the case when G is a connected semisimple Lie group or a motion group on the plane, all spherical functions are quasi-bounded (cf. [2]). A topologically irreducible representation which defines non-quasi-bounded spherical functions is called an *essentially non-Banach representation*. Here we give examples of essentially non-Banach representations of a semidirect product group $G = S \rtimes K$, where S is a free group with infinitely many generators and K is a compact abelian group.

§1. A semidirect product group $G = S \rtimes K$. We denote by N or Z the set of natural numbers or integers respectively. Let S be a free group with discrete topology generated by infinitely many generators s_n, $n \in N$. The automorphism group $\text{Aut}\langle s_n \rangle$ of the infinite cyclic group $\langle s_n \rangle = \{s_n^m \mid m \in Z\}$ consists of two elements, 1_n the identity and ψ_n the automorphism of $\langle s_n \rangle$ such that $\psi_n(s_n) = s_n^{-1}$. Let $K = \prod_{n \in N} \text{Aut}\langle s_n \rangle$ be the direct product group which is compact with respect to the product topology. Then K is naturally embedded into $\text{Aut}\, S$ as $k \cdot s = k_{n_1}(s_{n_1}^{m_1}) \cdots k_{n_p}(s_{n_p}^{m_p})$ for $k = (k_n) \in K$ and $s = s_{n_1}^{m_1} \cdots s_{n_p}^{m_p} (m_j \in Z, n_j \in N)$. The semidirect product group $G = S \rtimes K$ is locally compact and unimodular. In §2 we will construct K-finite topologically irreducible representations of G which are essentially non-Banach representations.

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Here we introduce some notations concerning the unitary dual $\hat{K}$ to K. Let $(\mathrm{Aut}\langle s_n\rangle)^\frown=\{1, \kappa_n\}$ be the unitary dual of $\mathrm{Aut}\langle s_n\rangle$, where 1 is the trivial character and κ_n is the one for which $\kappa_n(\psi_n)=-1$. Then $\hat{K}\cong\prod'_{n\in N}(\mathrm{Aut}\langle s_n\rangle)^\frown$, where the right hand side denotes the restricted direct product, namely, the set of all $\delta=(\delta_n)\in\prod_{n\in N}(\mathrm{Aut}\langle s_n\rangle)^\frown$ such that $\delta_n=1$ except for finitely many $n\in N$. Note that $\delta(k)=\prod_{n\in N}\delta_n(k_n)$ for $\delta=(\delta_n)\in\prod'_{n\in N}(\mathrm{Aut}\langle s_n\rangle)^\frown$ and $k=(k_n)\in K$. For each generator s_n of S we put

$$K(s_n)=\{k\in K\,|\,ks_nk^{-1}=s_n\}=\{k\in K\,|\,k_n=1_n\},$$

then this is a normal subgroup of K of index 2. Then two unitary characters $\sigma=(\sigma_n)$, $\tau=(\tau_n)\in\hat{K}$ are identical on $K(s_n)$ if and only if $\sigma_m=\tau_m$ for all $m\neq n$. Now we use the notation $\sigma\overset{n}{\rightarrow}\tau$ to indicate the situation that $\sigma_m=\tau_m$ for all $m\neq n$, $\sigma_n=1$ and $\tau_n=\kappa_n$. Whenever we choose a number $n\in N$, we can divide $\hat{K}$ into the collection of ordered pairs (σ,τ) with $\sigma\overset{n}{\rightarrow}\tau$. Another notation we need is $l(\sigma)=\#\{n\in N\,|\,3\leqq n,\ \sigma_n\neq 1\}$ for $\sigma=(\sigma_n)\in\hat{K}$.

§ 2. **Essentially non-Banach representations of G.** With every $\delta\in\hat{K}$ we associate an abstract element $v(\delta)$, and denote by $\mathfrak{H}$ the vector space of all finite linear combinations of $v(\delta)$'s with complex coefficients. Let C be the set of complex numbers. Into the vector space $\mathfrak{H}_F=\sum_{\delta\in F}Cv(\delta)$, where F is a finite subset of $\hat{K}$, we introduce a norm $\|v\|_F=\max_{\delta\in F}|c(\delta)|$ for $v=\sum_{\delta\in F}c(\delta)v(\delta)$. Then $\mathfrak{H}$ is a locally convex complete topological vector space as the inductive limit of $\mathfrak{H}_F$'s.

Let a be a non-zero constant. For s_1, the first member of generators of S, we define a continuous linear operator $T^a(s_1)$ on $\mathfrak{H}$ as follows. First we divide $\hat{K}$ into the collection of ordered pairs (σ,τ) with $\sigma\overset{1}{\rightarrow}\tau$. Using a conventional notation $a[m]=a^m$ for $m\in Z$ to avoid too much complicated indicies, we define the action of $T^a(s_1)$ on $Cv(\sigma)+Cv(\tau)$ as

(1) in case $\sigma\overset{1}{\rightarrow}\tau$ and $\sigma_2=\tau_2=1$,

$$T^a(s_1)v(\sigma)=a[l(\sigma)]v(\tau),\quad T^a(s_1)v(\tau)=-a[-l(\sigma)]v(\sigma),$$

(2) in case $\sigma\overset{1}{\rightarrow}\tau$ and $\sigma_2=\tau_2=\kappa_2$,

$$T^a(s_1)v(\sigma)=a[-l(\sigma)]v(\tau),\quad T^a(s_1)v(\tau)=-a[l(\sigma)]v(\sigma).$$

For other generators s_n $(n\geqq 2)$ we define the action of $T^a(s_n)$ on $Cv(\sigma)+Cv(\tau)$ for $\sigma\overset{n}{\rightarrow}\tau$,

(3) $T^a(s_n)v(\sigma)=v(\tau),\quad T^a(s_n)v(\tau)=-v(\sigma).$

All these operators $T^a(s_n)$ are invertible and $T^a(s_n)^{-1}$ are given in the following forms:

(1^{-1}) for $\sigma\overset{1}{\rightarrow}\tau$, $\sigma_2=\tau_2=1$,

$$T^a(s_1)^{-1}v(\sigma)=-a[l(\sigma)]v(\tau),\quad T^a(s_1)^{-1}v(\tau)=a[-l(\sigma)]v(\sigma),$$

(2^{-1}) for $\sigma\overset{1}{\rightarrow}\tau$, $\sigma_2=\tau_2=\kappa_2$,

$$T^a(s_1)^{-1}v(\sigma)=-a[-l(\sigma)]v(\tau),\quad T^a(s_1)^{-1}v(\tau)=a[l(\sigma)]v(\sigma),$$

(3^{-1}) for $\sigma\overset{n}{\rightarrow}\tau$ $(n\geqq 2)$,

$$T^a(s_n)^{-1}v(\sigma)=-v(\tau),\quad T^a(s_n)^{-1}v(\tau)=v(\sigma).$$

Therefore, putting $T^a(s_n^{-1})=T^a(s_n)^{-1}$, we obtain a representation $s \to T^a(s)$ of S on $\mathfrak{H}$ in the obvious way.

A continuous linear operator $T^a(k)$, $k \in K$, on $\mathfrak{H}$ is defined as $T^a(k)v(\delta) = \delta(k)v(\delta)$ for $\delta \in \hat{K}$. Then $k \to T^a(k)$ is a representation of K on $\mathfrak{H}$, and the vector subspace $\mathfrak{H}(\delta)$ of all vectors which are transformed according to δ under $T^a(k)$ $(k \in K)$ is just $\boldsymbol{C}v(\delta)$.

It is easily checked that we have $T^a(ks_nk^{-1})=T^a(k)T^a(s_n)T^a(k)^{-1}$ for every $k \in K$ and $n \in N$, and so $T^a(ksk^{-1})=T^a(k)T^a(s)T^a(k)^{-1}$ for all $k \in K$ and $s \in S$. By defining $T^a(x)=T^a(s)T^a(k)$ for $x=sk \in G=S \rtimes K$ with $s \in S$ and $k \in K$, we obtain a representation $T^a(x)$ of G on $\mathfrak{H}$.

Theorem. *The representations $\{\mathfrak{H}, T^a(x)\}$ of G are topologically irreducible for all $a \neq 0$, and are essentially non-Banach representations if and only if $|a| \neq 1$.*

Proof. Take any $\sigma=(\sigma_n) \in \hat{K}$ and denote by $n_1, \cdots, n_p$ $(1 \leqq n_1 < \cdots < n_p)$ the whole elements of the set $\{n \in N \mid \sigma_n = \kappa_n\}$. Then, by the definition of operators $T^a(s_n)$, it is clear that $T^a(s_{n_p} \cdots s_{n_1})v(1)=v(\sigma)$. This means that there exist no closed invariant non-trivial subspaces of $\mathfrak{H}$, namely, topological irreducibility of $\{\mathfrak{H}, T^a(x)\}$.

Assume that $\{\mathfrak{H}, T^a(x)\}$, for some $a \neq 0$, defines quasibounded spherical functions. Then they are also defined from a Banach representation $\{\mathfrak{B}, P(x)\}$, and there exists a linear bijection ξ of $\mathfrak{H}$ onto an invariant subspace $\mathfrak{B}_0$ of $\mathfrak{B}$ such that $\xi \circ T^a(x)=P(x) \circ \xi$ for all $x \in G$ [3]. Now for any $\sigma=(\sigma_n) \in \hat{K}$ such that $\sigma_1=1$ and $\sigma_2=1$ we have

$$P(s_2s_1s_2s_1)\xi(v(\sigma))=\xi(T^a(s_2s_1s_2s_1)v(\sigma))=a[2l(\sigma)]\xi(v(\sigma)),$$

therefore $\|P(s_2s_1s_2s_1)\| \geqq |a[2l(\sigma)]|$. Since we can take $\sigma \in \hat{K}$ for which $l(\sigma)$ is arbitrarily large, we have $|a| \leqq 1$. On the other hand, for any $\sigma \in \hat{K}$ such that $\sigma_1=\kappa_1$ and $\sigma_2=1$ we have $P(s_2s_1s_2s_1)\xi(v(\sigma))=a[-2l(\sigma)]\xi(v(\sigma))$, whence $|a| \geqq 1$. Thus $|a|=1$.

Conversely if $|a|=1$, it is easy to see that the corresponding spherical functions are bounded. Now the proof is completed.

§3. Explicit formula of the spherical functions of type 1. Let us introduce some notations. We denote by $|s|$ the length of $s \in S$. An element $s \in S$ is called a *power* of s_n if $s=s_n^m$ for some non-zero integer $m \in \boldsymbol{Z}$. If m is an odd (or even) integer, then $s=s_n^m$ is called an odd (or even, resp.) power of s_n. Suppose that s_n appears, taking its multiplicity into account, p times in the reduced expression of s and that s_n^{-1} does q times, then we say "s_n appears $p-q$ times altogether in s". We use this terminology even when $p=0$, $q=0$ or $p-q \leqq 0$. For any element $s \in S$ we pick up all generators $s_{n_1}, \cdots, s_{n_p}$ $(n_1 < n_2 < \cdots < n_p)$ which appear odd number of times altogether in s and put $s^*=s_{n_1} \cdots s_{n_p}$. If there exist no such generators, then we put $s^*=1$.

Now we give an explicit formula of the spherical function $\phi_1^a(x)$ of type 1 which is defined from the representation $\{\mathfrak{H}, T^a(x)\}$, $a \neq 0$, constructed in §2. Since $\phi_1^a(sk)=\phi_1^a(s)$, it is enough to give the values of ϕ_1^a on S. We give them according to cases as follows.

Case (1). If there exists a generator which appears odd number of times altogether in s, then

(E1) $$\phi_1^a(s)=0.$$

Case (2). Suppose that each s_n appears $2p_n$ times ($p_n \in Z$) altogether in s, and that odd powers of s_1 do not appear in the reduced expression of s, then

(E2) $$\phi_1^a(s)=\prod_{n\in N}(-1)^{p_n}.$$

Case (3). Suppose that each s_n appears $2p_n$ times ($p_n \in Z$) altogether in s, and that odd powers of s_1 appear in the reduced expression of s. Let the expression of s be $s=u_{r+1}\cdot s_1^{m_r}\cdot u_r\cdots u_2\cdot s_1^{m_1}\cdot u_1$, where $m_1,\cdots,m_r$ are odd integers and $u_1,\cdots,u_{r+1}$ have no odd powers of s_1 in their expressions. For $1\leqq i\leqq r$, we assume s_2 appears n_i times altogether in u_i. Let t_i ($1\leqq i\leqq r$) be the elements in S which come out from the expressions of u_i by getting rid of all powers of s_2. Then we have

(E3) $$\phi_1^a(s)=\prod_{n\in N}(-1)^{p_n}\prod_{i=1}^{r}a[(-1)^{i-1}(-1)^{n_1+\cdots+n_i}|(t_1\cdots t_i)^*|].$$

References

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