20. A Note on the Artin Map. III

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This is a continuation of our preceding papers [4], [5] which will be referred to as (I), (II), respectively. In (I), we defined, for a finite Galois extension K/k of number fields, a surjective monoid homomorphism (*a generalized Artin map*)

(0.1) $$\alpha_{K/k}: I^+(S) \longrightarrow M$$

where S is a finite set of finite primes of k containing all primes which ramify in K, $I^+(S)$ the free commutative monoid generated by primes $\mathfrak{p} \notin S$ and M the multiplicative monoid generated by averaged class sums (see (1.1)) in the center of the group ring $Z[1/n][G]$, $G=G(K/k)$, $n=$the order of G.[1] In this paper, we study the K-group $K(M)$ of the monoid M and prove that $K(M) \approx G^{ab}=G/G'$, G' being the commutator subgroup of G. Then, taking the $K(*)$ of both sides of (0.1), we get a surjective group homomorphism

(0.2) $$I(S) \longrightarrow G^{ab}$$

which is essentially the Artin map in the maximal abelian subextension K'/k of K/k.

§1. $K(M(G))$. Let G be a finite group and σ_i, $1 \leq i \leq r$, be a complete set of representatives of conjugate classes of G. Denoting by h_i the cardinality of the conjugate class containing σ_i, consider the averaged class sum:

(1.1) $$\gamma_i = \frac{1}{h_i} \sum_{\sigma \sim \sigma_i} \sigma \in Z\left[\frac{1}{n}\right][G], \qquad n=\text{order of } G,$$

and the commutative monoid $M(G)$ generated by γ_i, $1 \leq i \leq r$, in the center of $Z[1/n][G]$:

(1.2) $$M(G) = \langle \gamma_1, \cdots, \gamma_r \rangle.$$

Let $K(M(G))$ denote the K-group of $M(G)$.[2] The canonical surjective homomorphism $G \to G^{ab}$ extends, by linearity, to a ring homomorphism $Z[1/n][G] \to Z[1/n][G^{ab}]$, which induces a surjective monoid homomorphism

(1.3) $$\varphi: M(G) \longrightarrow M(G^{ab}) = G^{ab}$$

such that $\varphi(\gamma_i) = \sigma_i \bmod G'$.[3] Therefore there exists a surjective homomorphism $\bar{\varphi}$ which makes the diagram below commutative:

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1) Unlike (I), (II), we use $I^+(S)$ for monoids of integral ideals and use $I(S)$ for groups of fractional ideals, i.e., $I(S)=K(I^+(S))$. I take this opportunity to thank Messrs. Morishita and Tanabe for discussions on K-groups.

2) As for the K-group (or the Grothendieck group) of monoids, see, e.g., [2], pp. 58–59.

3) Note that we took the average of the class sum in (1.1).

(1.4)
$$\begin{array}{ccc} M(G) & \xrightarrow{\kappa} & K(M(G)) \\ & \underset{\varphi}{\searrow} \quad \underset{\bar{\varphi}}{\swarrow} & \\ & G^{ab} & \end{array}$$

If we adopt, for a commutative monoid M, the mode of definition

(1.5)
$$K(M) \overset{\text{def}}{=} (M\times M)/\sim$$

where, for $a, b, c, d \in M$,

$$(a, b)\sim(c, d) \overset{\text{def}}{\Longleftrightarrow} adu=bcu \qquad \text{for some } u\in M$$

and put $[a, b]=$class of (a, b) in $K(M)$, then we have $\kappa(a)=[a, 1]$, $\bar{\varphi}([a, b])=\varphi(a)\varphi^{-1}(b)$ in (1.4).

Now, back to our $M=M(G)$, we want to study $\operatorname{Ker}\bar{\varphi}$. To do this, we had better go to $\boldsymbol{C}$-characters of G. Let χ_ν, $1\leqq\nu\leqq r$, be the set of all irreducible $\boldsymbol{C}$-characters of G. Among them we agree that χ_ν, $1\leqq\nu\leqq s$, are linear and the rest are nonlinear. For each character χ, we put $\chi^*(\sigma)=\chi(\sigma)/\chi(1)$, $\sigma\in G$. Let $\boldsymbol{C}[G]_0$ be the center of the group ring $\boldsymbol{C}[G]$. Then by the isomorphism

(1.6)
$$\omega: \boldsymbol{C}[G]_0 \xrightarrow{\sim} \boldsymbol{C}^r,$$

we can identify $M=\langle\gamma_i\rangle$ with the monoid $X=\langle\xi_i\rangle$ where

(1.7)
$$\xi_i=\omega(\gamma_i)=\begin{pmatrix} \vdots \\ \chi_\nu^*(\sigma_i) \\ \vdots \end{pmatrix}\in \boldsymbol{C}^r.\text{[4)]}$$

For two elements $a, b\in M(G)$, suppose that

(1.8)
$$a=\prod_{i=1}^r \gamma_i^{e_i}, \qquad b=\prod_{i=1}^r \gamma_i^{f_i}.$$

Then we have

$$\begin{aligned}[a, b]\in \operatorname{Ker}\bar{\varphi} &\Longleftrightarrow \varphi(a)\varphi^{-1}(b)=1 \\ &\Longleftrightarrow \prod_{i=1}^r \sigma_i^{e_i}\Big(\prod_{i=1}^r \sigma_i^{f_i}\Big)^{-1}\in G' \\ &\Longleftrightarrow \prod_{i=1}^r \chi_\nu(\sigma_i)^{e_i-f_i}=1, \qquad 1\leqq\nu\leqq s,\end{aligned}$$

and so

(1.9)
$$[a, b]\in \operatorname{Ker}\bar{\varphi} \Longleftrightarrow \prod_{i=1}^r \chi_\nu(\sigma_i)^{e_i}=\prod_{i=1}^r \chi_\nu(\sigma_i)^{f_i}, \qquad 1\leqq\nu\leqq s.$$

Now put $u=\prod_{i=1}^r\gamma_i$. Since (1.7) yields

$$\omega(a)=\prod_{i=1}^r \xi_i^{e_i}, \qquad \omega(b)=\prod_{i=1}^r \xi_i^{f_i},$$

we have

$$\omega(au)=\prod_{i=1}^r \xi_i^{e_i+1}=\begin{bmatrix} \vdots \\ \prod_{i=1}^r \chi_\nu(\sigma_i)^{e_i+1} \\ \vdots \\ \prod_{i=1}^r \chi_\nu^*(\sigma_i)^{e_i+1} \\ \vdots \end{bmatrix} \begin{array}{l} \\ \nu\leqq s, \\ \\ \nu>s. \\ \\ \end{array}$$

4) See (I), (7)-(10), also (II), (2.7), (2.8).

By a theorem of Burnside ([1], p. 36, (6.9)), for $\nu > s$, we have $\chi_\nu^*(\sigma_i)=0$ for some i, $1 \leqq i \leqq r$. Hence we get

$$(1.10) \qquad \omega(au)=\begin{pmatrix} \vdots \\ \prod_{i=1}^{r} \chi_\nu(\sigma_i)^{e_i+1} \\ \vdots \\ 0 \\ \vdots \end{pmatrix}, \quad \text{and similarly for } bu.$$

Therefore, if $[a, b] \in \operatorname{Ker} \bar{\varphi}$, we have, from (1.9), (1.10), $\omega(au)=\omega(bu)$ and so $au=bu$, i.e., $[a, b] \sim [1, 1]$, which shows that $\operatorname{Ker} \bar{\varphi}=1$. Since $\bar{\varphi}$ is surjective, we get the isomorphism:

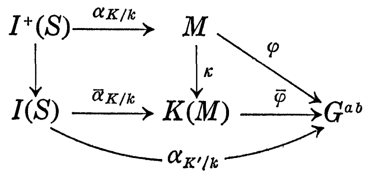
$$(1.11) \qquad \bar{\varphi}: K(M(G)) \xrightarrow{\sim} G^{ab}.$$

§ 2. $\boldsymbol{\alpha_{K/k}}$ and $\boldsymbol{\alpha_{K'/k}}$. Taking the K-groups of both sides of (0.1), we obtain a surjective group homomorphism

$$(2.1) \qquad \bar{\alpha}_{K/k}: I(S) \longrightarrow K(M).$$

On the other hand, let K' be the maximal abelian subextension of K/k. Then, we have the Artin map (for abelian extensions):

$$\alpha_{K'/k}: I(S) \longrightarrow G(K'/k)=G^{ab}, \qquad G=G(K/k).$$



For $\mathfrak{p} \in I(S)$, we have

$$\bar{\varphi}\bar{\alpha}_{K/k}(\mathfrak{p})=\varphi(\alpha_{K/k}(\mathfrak{p}))=\left[\frac{K/k}{\mathfrak{P}}\right] \bmod G'=\left[\frac{K'/k}{\mathfrak{P}}\right]=\left(\frac{K'/k}{\mathfrak{p}}\right)=\alpha_{K'/k}(\mathfrak{p}),$$

which shows that

$$(2.2) \qquad \bar{\varphi}\bar{\alpha}_{K/k}=\alpha_{K'/k}.$$

Remark. (2.2) implies that $\operatorname{Ker} \bar{\alpha}_{K/k}=\operatorname{Ker} \alpha_{K'/k}=S(\mathfrak{m})N_{K'/k}I_{K'}(\mathfrak{m})$, the Takagi group for suitable module $\mathfrak{m}$ in k.[5]

References

[1] Feit, W.: Characters of Finite Groups. Benjamin, New York-Amsterdam (1967).
[2] Hilton, P.: General Cohomology Theory and K-Theory. London Math. Soc. Lecture Note Series 1, Cambridge Univ. Press (1971).
[3] Iyanaga, S.: Number Theory. North-Holland, Amsterdam (1975).
[4] Ono, T.: A note on the Artin map. Proc. Japan Acad., **65**A, 304–306 (1989).
[5] ——: A note on the Artin map. II. ibid., **66**A, 132–136 (1990).

[5] See, e.g., [3].