17. Noetherian Property of Inductive Limits of Noetherian Local Rings

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The purpose of this note is to show the following,

Theorem 1. *Let* $\{(A_\lambda, m_\lambda) \mid \lambda \in \Lambda\}$ *be a filtered inductive system of noetherian local rings such that* $m_\lambda A_\mu = m_\mu$ *for* $\mu \geq \lambda$. *Then the inductive limit* A *of the system is noetherian.*

The motivation of proving Theorem 1 came from the lecture by Professor Nagata at Symposium on Commutative Algebra in Gifu. That is, he mentioned the following, which got to have been known (cf. [2, ch. 0_{III} Lemma 10.3.1.3], [1, ch. *III* § 5 exerc. 7]).

Theorem. *Let* $\{(A_i, m_i) \mid i \in N\}$ *be a chain of noetherian local rings such that for any* $i \in N$

1) $m_i A_{i+1} = m_{i+1}$
2) A_{i+1} *is flat over* A_i.

Then the union $\cup A_i$ *is a noetherian local ring.*

If we drop the assumption 1) in the Theorem, then we easily construct a system such that the union $\cup A_i$ is non-noetherian. He asked what about the assumption 2)?

Since there is a system $\{(A_i, m_i)\}$ of noetherian local rings with non-flat morphisms such that $m_i A = m$ with A non-noetherian where $A = \cup A_i$ and $m = \cup m_i$, (cf. [3, Appendix A1] and [4, § 2]), 2) might seem to be necessary. But we shall show that 1) is sufficient.

Now we begin with a lemma.

Lemma 2. *Let* (A, m) *be a quasi-local ring such that* m *is finitely generated, and let* I *be an ideal of* A. *Then the completion* $\widehat{(A/I)}$ *of* A/I *is isomorphic to* $\hat{A}/I\hat{A}$ *with the completion* $\hat{A}$ *of* A.

Proof. Since $\hat{A}$ is noetherian [3, (31.7) Corollary], $I\hat{A}$ is closed in $\hat{A}$ by [3, (16.7) Theorem] and $\hat{A}/I\hat{A}$ is a complete local ring. On the other hand, since $(\hat{A}/I\hat{A})/m^n(\hat{A}/I\hat{A}) = \hat{A}/I\hat{A} + m^n\hat{A} = A/I + m^n$, we have $\widehat{(A/I)} = \widehat{(\hat{A}/I\hat{A})} = \hat{A}/I\hat{A}$ by the projective limit characterization of completion.

Proposition 3. *Under the assumption of the Theorem* 1 *the natural homomorphism* $\varphi : A \to \hat{A}$ *to the completion* $\hat{A}$ *is injective.*

Proof. Let $k_\lambda = A_\lambda / m_\lambda$ be the residue field of A_λ, then $\{k_\lambda \mid \lambda \in \Lambda\}$ forms

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a filtered inductive system with inductive limit, say k, which is isomorphic to A/m.

Then $gr_{m_\lambda}A_\lambda \to gr_{m_\mu}A_\mu$ $(\mu \geq \lambda)$ induces a surjective homomorphism $gr_{m_\lambda}A_\lambda \otimes_{A_\lambda} k \to gr_{m_\mu}A_\mu \otimes_{A_\mu} k$ because $gr_{m_\lambda}A_\lambda \otimes_{A_\lambda} A_\mu \to gr_{m_\mu}A_\mu$ is surjective.

Thus the system $\{gr_{m_\lambda}A_\lambda \otimes_{A_\lambda} k \mid \lambda \in \Lambda\}$ forms a filtered inductive system of noetherian graded rings whose morphisms are surjective. Thus there exists a μ_0 such that for any $\lambda \geq \mu_0$, $gr_{m_\lambda}A_\lambda \otimes_{A_\lambda} k \to gr_m A$ is an isomorphism because of the noetherian property. In particular $gr_{m_\lambda}A_\lambda \to gr_m A$ is injective.

Let x be an element of A such that $\varphi(x)=0$. We may assume x is represented by an element y of A_λ. We may assume $\lambda \geq \mu_0$.

Then we have $y=0$. Really, otherwise, then there exists an integer n such that $y \in m_\lambda^n \setminus m_\lambda^{n+1}$ because A_λ is noetherian. Now the image by $gr_{m_\lambda}A_\lambda \to gr_m A$ of the class of y in $m_\lambda^n/m_\lambda^{n+1}$ is not zero, which means the class of x in m^n/m^{n+1} is not zero. But this contradicts the assumption $\varphi(x)=0$. Thus $y=0$ and φ is injective.

Proof of the Theorem 1. Let I be an ideal of A. Since $\hat{A}$ is noetherian, $I\hat{A}$ is finitely generated, say $I\hat{A}=J\hat{A}$, with an ideal J of A_{λ_0}.

We will show that $I=JA$. To see this, consider the inductive system $\{A_\lambda/JA_\lambda \mid \lambda \in \Lambda, \lambda \geq \lambda_0\}$ with the inductive limit A/JA. Now $(\widehat{A/JA})=\hat{A}/J\hat{A}$ by Lemma 2 and I/JA is contained in the kernel of the natural homomorphism $A/JA \to (\widehat{A/JA})$ because $(I/JA)(\widehat{A/JA})=I\hat{A}/J\hat{A}=0$.

On the other hand, this map is injective by Proposition 3 applying the system $\{(A_\lambda/JA_\lambda, m_\lambda/JA_\lambda)\}$, and we have $I/JA=0$, that is, $I=JA$.

References

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